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Guided Elastic Waves in a Pre-Stressed Hyperelastic Plate

A.Delory[1,2], F.Lemoult[1], A.Eddi[2], C.Prada[1]

GdR MecaWave – From 4th to 8th October 2021

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Back to Lamb Modes in a Soft Plate

Soft Solids

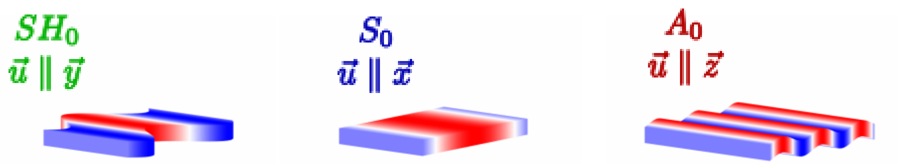
EcoFlex® = silicon gel

Order of magnitude : $v_T \sim 5$ m.s^{-1}
$v_L \sim 1000$ m.s^{-1}

Lamb modes in a Plate of Ecoflex

Families of modes

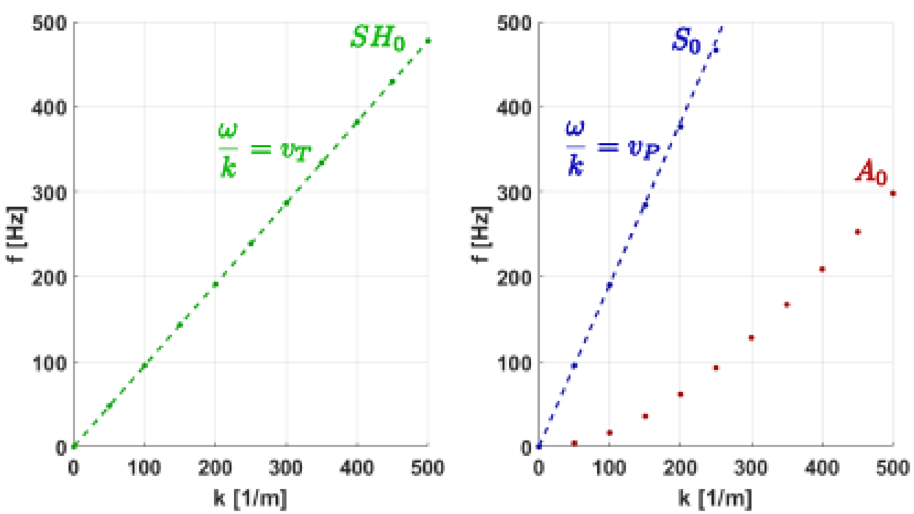
Unchanged SH
Symmetric S
Antisymmetric A



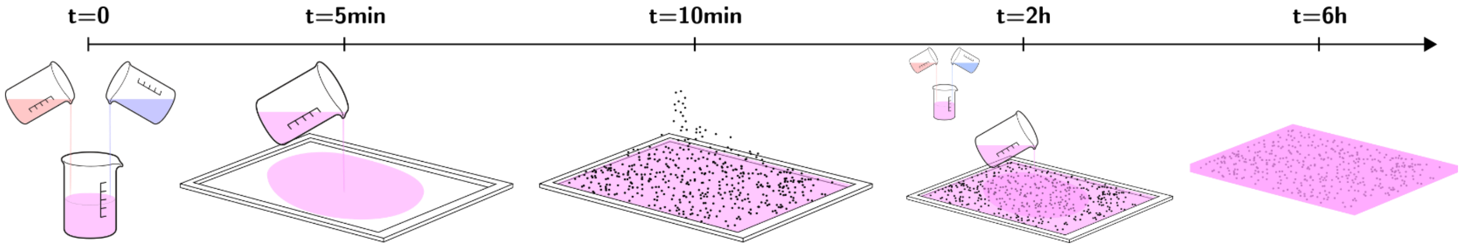
$v_P = 2v_T$

In-plane displacement & Non-dispersive

Dispersion Relation, thickness=3mm (COMSOL)



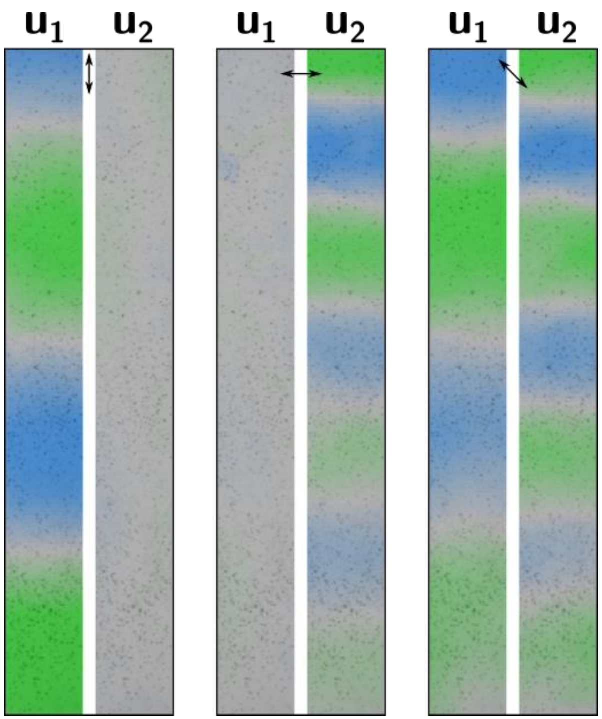
Experimental Setup



1. Prepare a plate
2. Monochromatic point source
3. Stroboscopy
4. Digital Image Correlation

Measurement of Mode Velocities in a Soft Plate

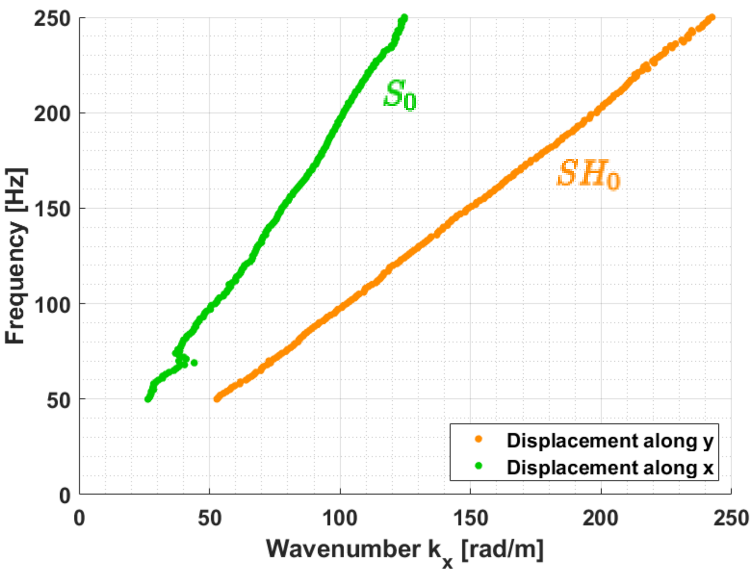
Monochromatic plane wave



Time-Fourier coefficient of the displacement field for different source polarisations at 120 Hz

Spatial FT

Mode dispersion and velocities

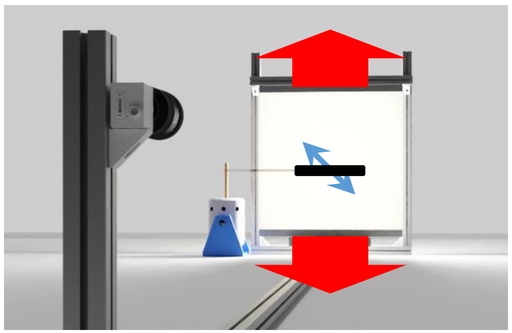


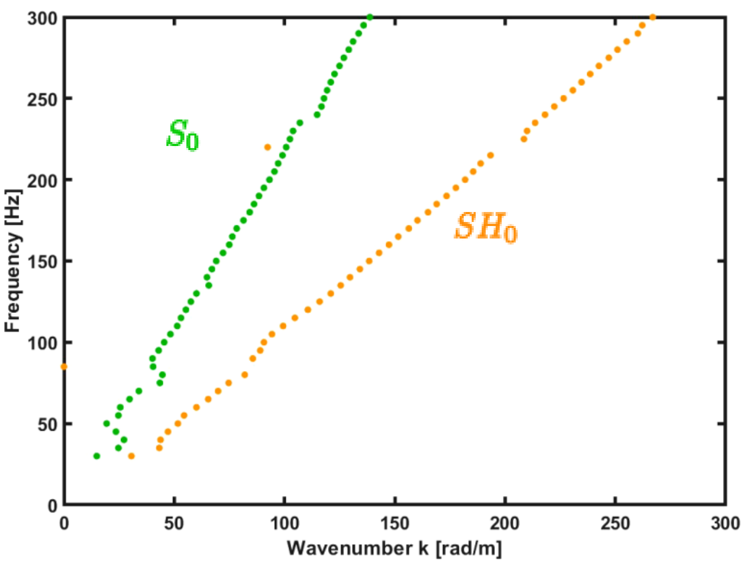
$v_T = 6\ \mathrm{m.s^{-1}}$

$v_P = 12\ \mathrm{m.s^{-1}}$

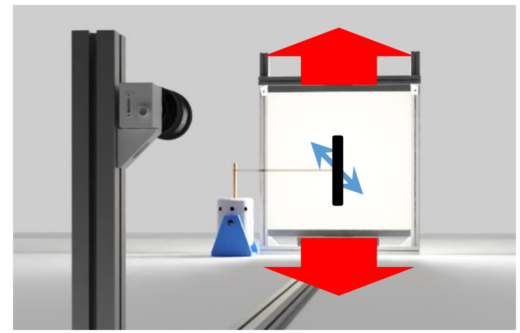
And in a Stretched Soft Plate

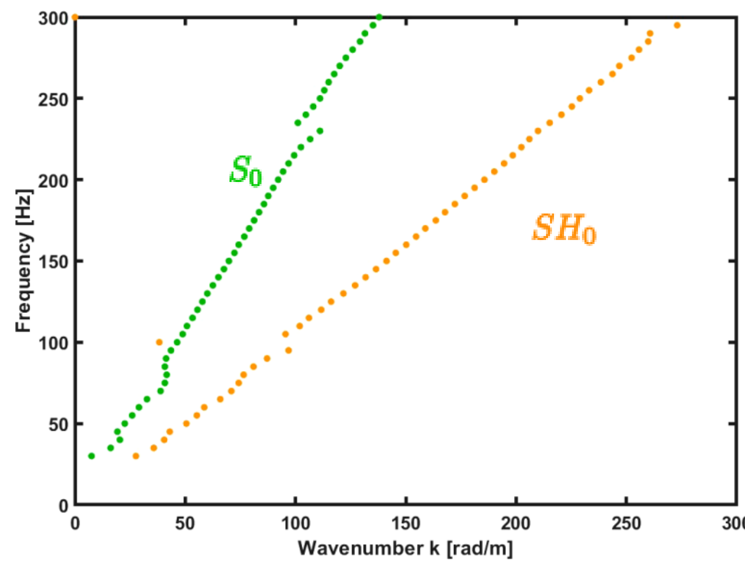
Along the elongation



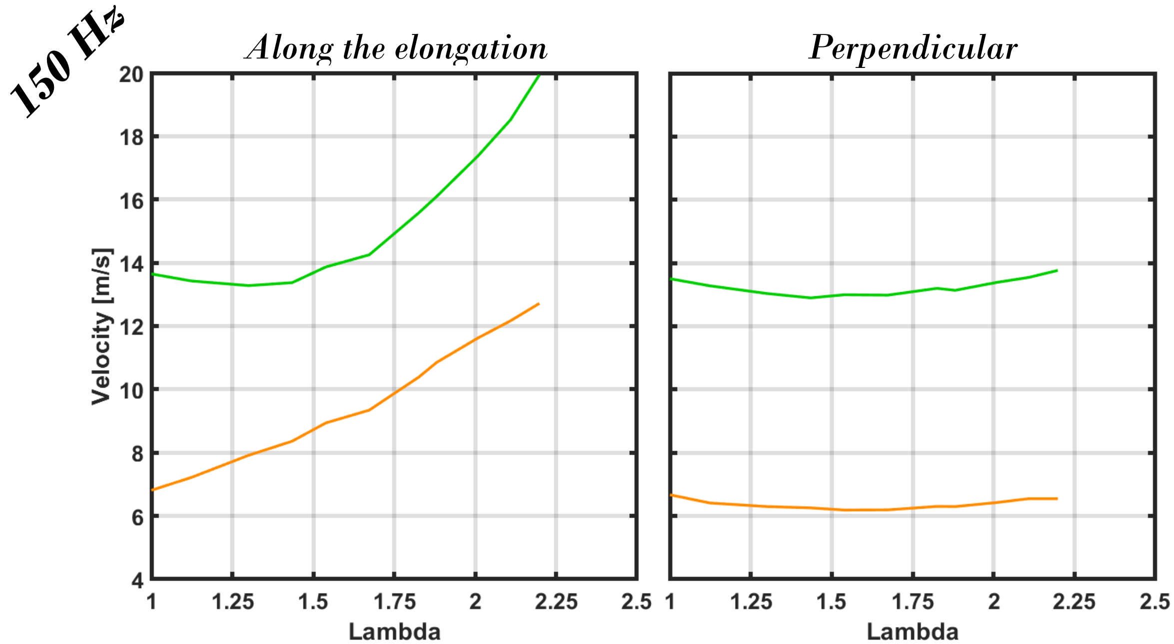


Perpendicular to the elongation





Evolution of Mode Velocities with Stretch Ratio



- Change in Velocities
- Non-linear Evolution
- Induced Anisotropy

How to build a model to understand those evolutions ?

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From Linear Elasticity …

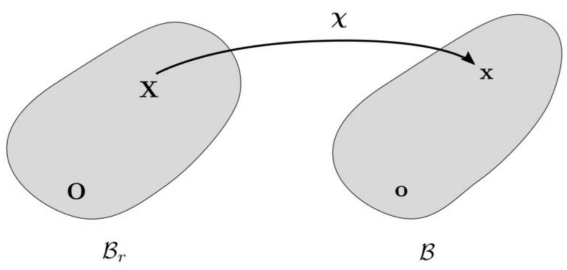
Let's define some useful tensors

Deformation gradient $\boldsymbol{F} = \text{Grad}\ \chi(\boldsymbol{X})$

Displacement field $\boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{x} - \boldsymbol{X}$

Linear Strain tensor $\epsilon = \dfrac{\nabla \mathbf{u}^{\top} + \nabla \mathbf{u}}{2}$

Cauchy's Stress tensor $\boldsymbol{\sigma}$



Hooke's Law = Constitutive Law for Linear Materials

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

Elasticity Tensor

where $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$

Major Symmetries → **Voigt Notation**

Wave Equation

$$C_{ijkl}\frac{\partial^2 u_l}{\partial X_j \partial X_k} = \rho_r \frac{\partial^2 u_i}{\partial t^2}$$

… to Non-Linear Elasticity

Neo-Hookean	$\frac{\mu}{2}\left(\bar{I}_1 - 3\right) + \frac{K}{2}\left(J-1\right)^2$
Mooney-Rivlin	$c_1\left(\bar{I}_1 - 3\right) + c_2\left(\bar{I}_2 - 3\right) + \frac{K}{2}\left(J-1\right)^2$
Gent	$-\frac{\mu J_m}{2}\ln\left(1 - \frac{\bar{I}_1 - 3}{J_m}\right) + \frac{K}{2}\left(J-1\right)^2$
Fung	$\frac{\mu}{2b}\left[e^{b(\bar{I}_1 - 3)} - 1\right] + \frac{K}{2}\left(J-1\right)^2$

Tensors need to be modified

$$\epsilon = \frac{\nabla \mathbf{u}^\top + \nabla \mathbf{u}}{2} \rightarrow \mathbf{E} = \frac{\nabla_0 \mathbf{u}^\top + \nabla_0 \mathbf{u}}{2} + \frac{\nabla_0 \mathbf{u}^\top \nabla_0 \mathbf{u}}{2}$$

Cauchy stress tensor defined in the deformed configuration $\boldsymbol{\sigma} \rightarrow \boldsymbol{S}$ 2nd Piola-Kirchhoff tensor defined in the reference configuration

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \rightarrow \mathbf{S} = f\left(\mathbf{E}\right)$$

Hyperelasticity, with a strain energy density function

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} \rightarrow \boxed{C_{ijkl} = \frac{\partial^2 W}{\partial F_{ji} \partial F_{lk}}}$$

and $C_{0ijkl} = \frac{1}{J} F_{ip} F_{kq} C_{pjql}$

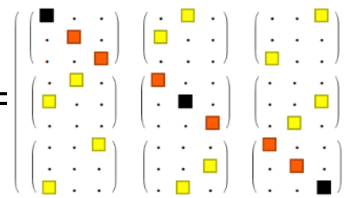
Eulerian Elasticity
(using material coordinates)

Wave Equation

$$C_{ijkl}\frac{\partial^2 u_l}{\partial X_j \partial X_k} = \rho_r \frac{\partial^2 u_i}{\partial t^2} \rightarrow \boxed{C_{0ijkl}\frac{\partial^2 u_l}{\partial x_j \partial x_k} = \rho \frac{\partial^2 u_i}{\partial t^2}}$$

Hyperelasticity

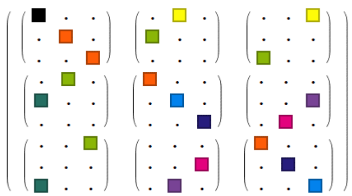
$$C_{ijkl} = \begin{pmatrix} \begin{pmatrix} c11 & 0 & 0 \\ 0 & c12 & 0 \\ 0 & 0 & c12 \end{pmatrix} & \begin{pmatrix} 0 & c66 & 0 \\ c66 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & c66 \\ 0 & 0 & 0 \\ c66 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & c66 & 0 \\ c66 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} c12 & 0 & 0 \\ 0 & c11 & 0 \\ 0 & 0 & c12 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c66 \\ 0 & c66 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & c66 \\ 0 & 0 & 0 \\ c66 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c66 \\ 0 & c66 & 0 \end{pmatrix} & \begin{pmatrix} c12 & 0 & 0 \\ 0 & c12 & 0 \\ 0 & 0 & c11 \end{pmatrix} \end{pmatrix} =$$



hyperelastic

eulerian

$$C_{0ijkl} =$$



Neo-Hookean	$\frac{\mu}{2}\left(\bar{I}_1 - 3\right) + \frac{K}{2}\left(J-1\right)^2$
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Gent	$-\frac{\mu J_m}{2} \ln\left(1 - \frac{\bar{I}_1 - 3}{J_m}\right) + \frac{K}{2}\left(J-1\right)^2$
Fung	$\frac{\mu}{2b}\left[e^{b(\bar{I}_1 - 3)} - 1\right] + \frac{K}{2}\left(J-1\right)^2$

Shear Velocities and Slowness Curves

Christoffel Equation for plane waves

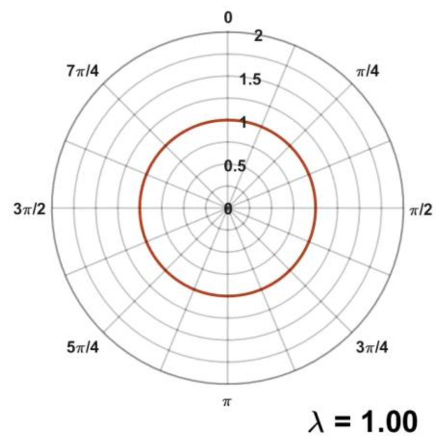
$$\rho V^2 u_i^0 = C_{ijkl}\, n_j n_k\, u_l^0$$

velocit
y

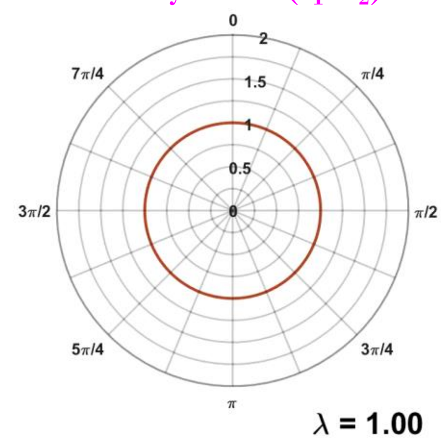
polarisatio
n

direction of
propagation

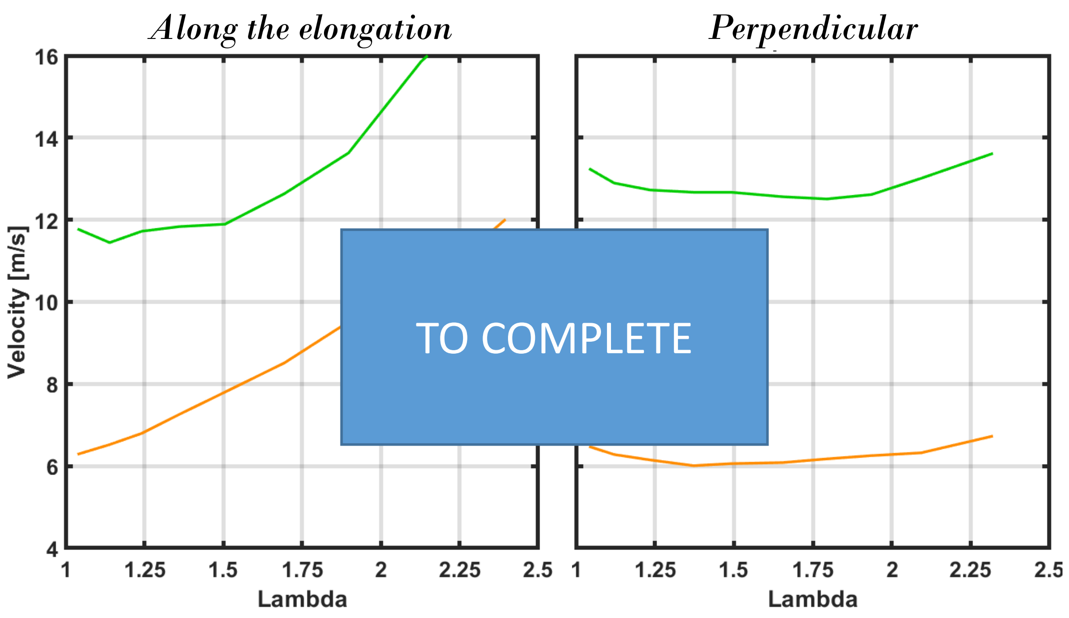
Neo-Hookean



Mooney-Rivlin (c_1=c_2)

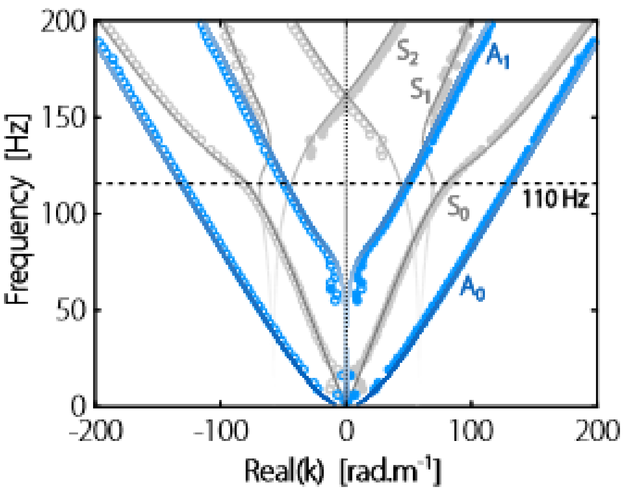


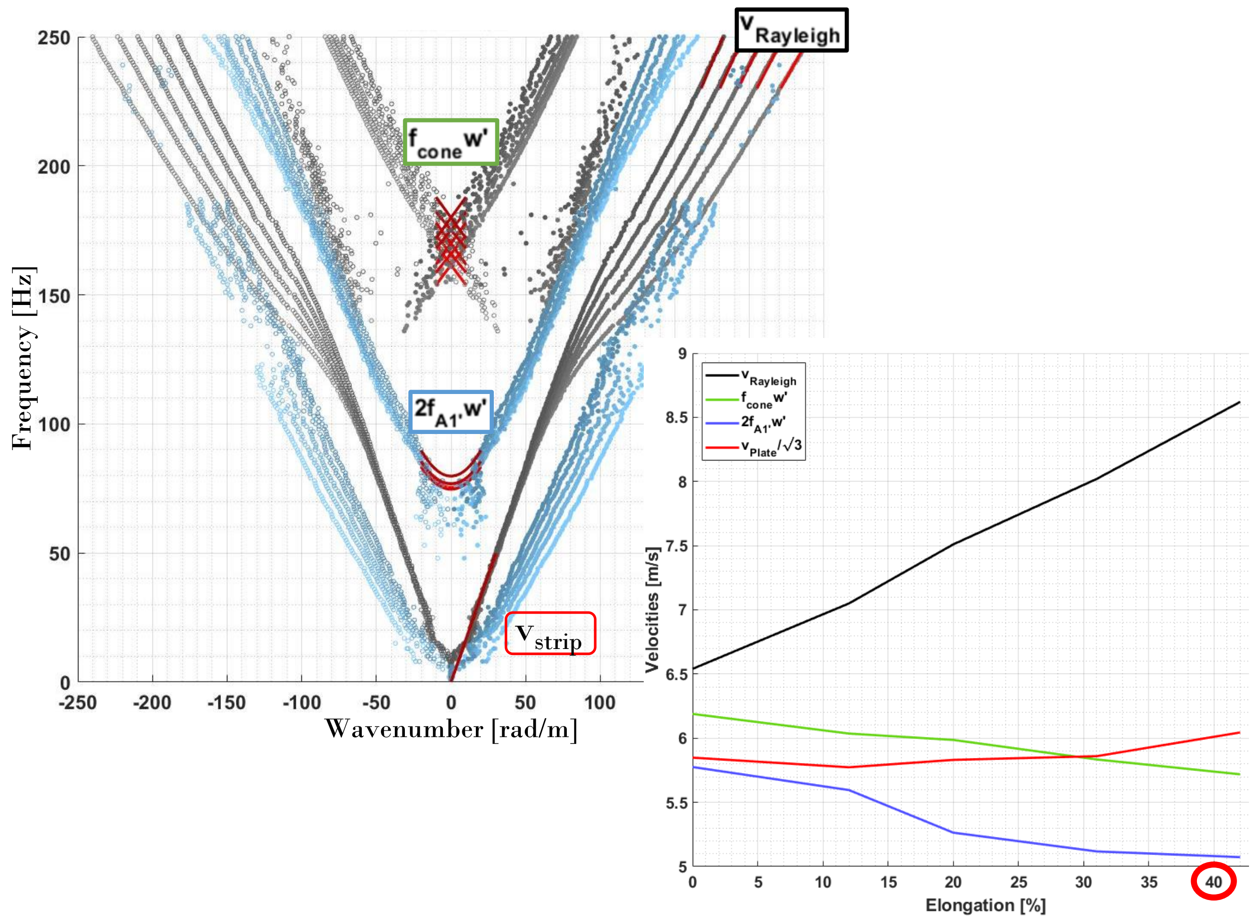
Fitting of Experimental Measurements



What's next ? Guided elastic waves in a soft strip

Dispersion relation of Lamb modes with new Poisson's ratio $\nu = 1/3$





J. Laurent, D. Royer, C. Prada. *In-plane backward and zero group velocity guided modes in rigid and soft strips.* JASA (2020)
M. Lanoy, F. Lemoult, A. Eddi, C. Prada. *Dirac cones and chiral selection of elastic waves in a soft strip*. PNAS 2020

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Thanks you for your attention !

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Rayleigh-Lamb Equation

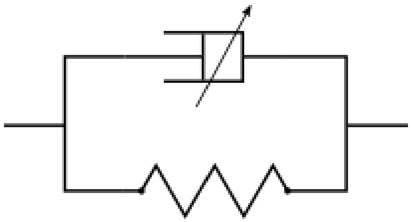
$$(k^2 - q^2)^2 \cos(pw + \alpha) \sin(qw + \alpha) + 4k^2 pq \sin(pw + \alpha) \cos(qw + \alpha) = 0$$

$$\text{Where} \begin{cases} p^2 = (\omega/v_L)^2 - k^2 \\ q^2 = (\omega/v_T)^2 - k^2 \\ \alpha = 0 \text{ for } S \text{ and } \alpha = \pi/2 \text{ for } A \end{cases}$$

D. Royer, E. Dieulesaint. *Elastic Waves in Solids I*, Masson 1996

Rheology of Ecoflex

Fractionnal Kelvin-Voigt
Viscoelastic Model



$$\mu = \mu(\omega) = \mu_0\left(1 + (i\omega\tau)^{\alpha}\right) \quad \text{with} \begin{cases} \mu_0 & = 25 \text{ kPa} \\ \tau & = 0.23 \text{ ms} \\ \alpha & = 0.3 \end{cases}$$