Hyperbolic model of bubbly fluids: variational approach

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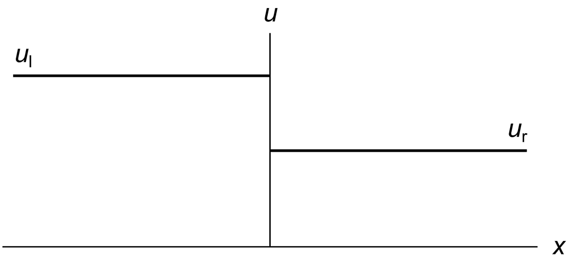
Outline

- What is a shock wave? dispersion? a dispersive shock wave?
- Dispersive vs hyperbolic: the classic shallow water model example
- Hamilton's principle: how to derive physically correct models?
- Iordanskii-Wijngaarden-Kogarko model of bubbly fluids
- Hyperbolization procedure: the augmented Lagrangian concept
- Numerical results

Systems of PDEs in conservative form (non-linear and usually hyperbolic)

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U})$$

Classic example: Initial value problem for Hopf equation with discontinuous initial data



$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$
$$u(x,0) = \begin{cases} u_l, & x \leq 0, \\ u_r, & x > 0. \end{cases}$$

Dispersion

Dispersion relation:

$$c_p = c_p(k)$$

Dispersive shock waves

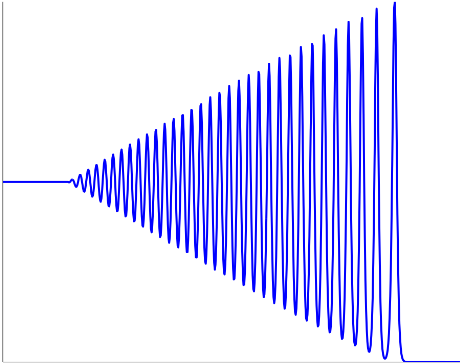


Figure: Dispersive shock wave in KdV equation

Korteweg-de Vries equation:

$$u_t + \left(\frac{u^2}{2} + u_{xx}\right)_x = 0,$$

Hyperbolic vs dispersive

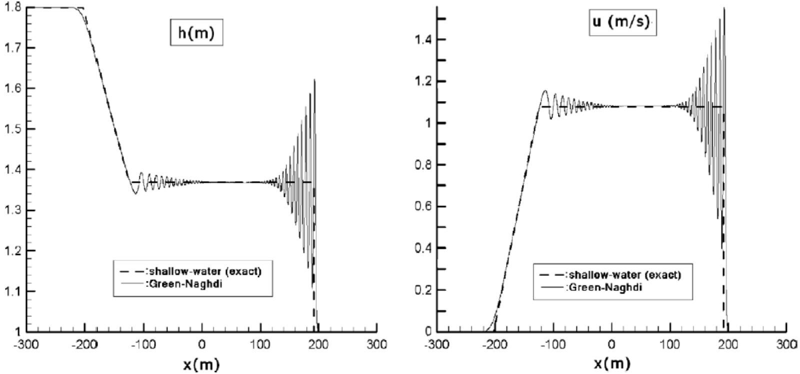


Figure: Shallow water vs Green-Naghdi (O. Le Métayer et al., 2010).

Shallow water and Green-Naghdi equations:

$$\begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + (hu^2 + p)_x &= 0 \end{aligned} \qquad p = \frac{gh^2}{2} + \frac{1}{3}h^2\ddot{h}, \qquad \dot{(\,)} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

Variational approach: classic exemple

Aristotle (384–322 B.C.):

Nature follows the easiest path...

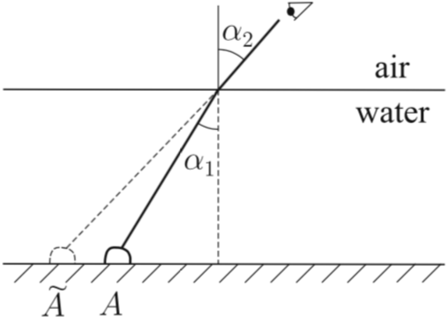


Figure: Fermat's principle: Time $\rightarrow$ *min*

Variational approach: Hamilton's principle in mechanical systems

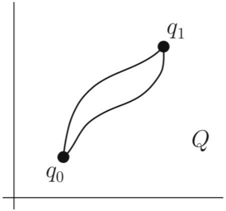


Figure: Admisible trajectories in the configurational space

The Lagrangian:

$$\mathcal{L}(q, \dot{q}, t) = \mathcal{K}(q, \dot{q}, t) - \mathcal{W}(q, t),$$

The action functional:

$$a = \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}, t) dt.$$

Hamiltonian's principle of stationary action

The true motion of a mechanical system is the stationary point of the functional

$$a = \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}, t)dt.$$

on the set of all paths beginning at point q_0 and instant t_0 and ending at point q_1 at instant t_1:

$$\delta a = 0. \tag{1}$$

Morover, if the Lagrangian $\mathcal{L}$ doesn't depend explicitely on time:

$$\mathcal{L} = \mathcal{L}(q, \dot{q})$$

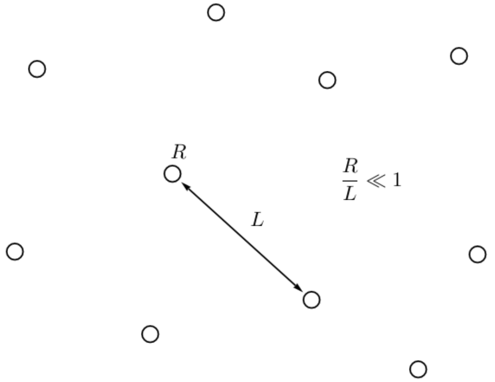
then the total energy conservation is guaranteed.

The equation (1) is equivalent to the correspondent Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = 0.$$

Bubbly fluids

What is a bubbly fluid?



Fluid kinetic energy due to oscillations of a single bubble of compressible gaz (Iordanski-Wijngaarden-Kogarko (1960s)):

$$2\pi\rho_l R^3 \dot{R}^2$$

Bubbly fluids:

Mixture density:

$$\rho = \alpha_l \rho_l + \alpha_g \rho_g$$

Total energy:

$$E = \frac{\rho |\mathbf{u}|^2}{2} + 2\pi N \rho_l R^3 \dot{R}^2 + \rho Y_g \varepsilon_g(\rho_g),$$

The Lagrangian of a bubbly fluid:

$$\mathcal{L} = \int_{\Omega(t)} \left(\frac{\rho |\mathbf{u}|^2}{2} + 2\pi N \rho_l R^3 \dot{R}^2 - \rho Y_g \varepsilon_g(\rho_g) \right) d\Omega,$$

Euler-Lagrange equations for the bubbly fluid model:

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0,$$
$$(\rho \mathbf{u})_t + \operatorname{div}(h\mathbf{u} \otimes \mathbf{u} + p) = 0.$$

$$p = p_g - \rho_l (R\ddot{R} + \frac{3}{2}\dot{R}^2)$$

Energy conservation:

$$\frac{\partial E}{\partial t} + \operatorname{div}\left(E\mathbf{u} + p\mathbf{u} \right) = 0.$$

Why is this model so good?

- It admits the variational formulation
- It is Galileian galilean invariant
- The total energy conservation is respected
- It describes the behaviour of bubbly fluids very well

But what's wrong with it?

- It's not hyperbolic
- It doesn't admit discontinuous solutions
- Enormous calculation time
- Mess with boundary conditions

The augmented Lagrangian concept

The relaxation variable η:

$$\mathcal{L}(\mathbf{u}, \rho, \dot{\rho}) \quad \Longrightarrow \quad \mathcal{L}(\mathbf{u}, \rho, \eta, \dot{\eta})$$

The density ρ is no more responsible for dispersive effects.
The relaxed variable η will tend to ρ in a certain limit.

Bubbly fluids augmented model

The bubble radius R can be expressed in terms of ρ:

$$R^3 = \frac{3}{4\pi n}\left(\frac{1}{\rho} - \frac{Y_1}{\rho_{10}}\right).$$

Original bubbly fluid Lagrangian:

$$\mathcal{L} = \int_{\Omega(t)} \left(\frac{\rho|\mathbf{u}|^2}{2} + 2\pi N\rho_l R^3\dot{R}^2 - \rho Y_g\varepsilon_g(\rho_g)\right) d\Omega.$$

Extended Lagrangian:

$$\hat{\mathcal{L}} = \int_{\Omega(t)} \left(\frac{\rho|\mathbf{u}|^2}{2} + 2\pi N\rho_l\dot{\eta}^2 - \rho Y_g\varepsilon_g(\rho_g) - \frac{\lambda\rho}{2}\left(\frac{f(\eta)}{\rho} - 1\right)^2\right) d\Omega.$$

Relaxation variable η:

$$\lambda \to \infty \quad \Longrightarrow \quad f(\eta) \to \rho, \quad R^3\dot{R}^2 \to \dot{\eta}^2$$

Bubbly fluids augmented model

The Euler-Lagrange equations corresponding to the extended Lagrangian:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = 0,$$
$$\frac{\partial \rho \mathbf{u}}{\partial t} + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u} + pI) = 0,$$
$$\frac{\partial \rho \eta}{\partial t} + \mathrm{div}(\rho \eta \mathbf{u}) = \rho \zeta$$
$$\frac{\partial \rho \zeta}{\partial t} + \mathrm{div}(\rho \zeta \mathbf{u}) = -\frac{\lambda}{\beta} \left(\frac{f(\eta)}{\rho} - 1 \right) f'(\eta),$$

where

$$p = p_g - \lambda f(\eta) \left(\frac{f(\eta)}{\rho} - 1 \right),$$

The model is unconditionnaly hyperbolic:

$$\frac{\partial p}{\partial \rho} > 0.$$

Numerical results

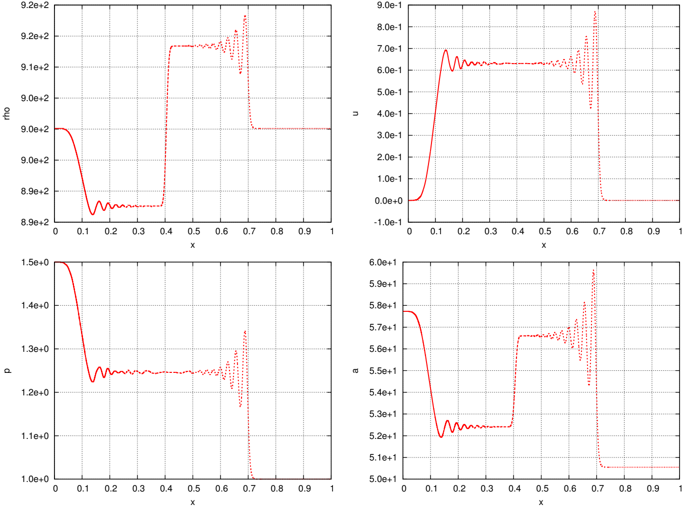


Figure: Riemann problem for Augmented bubbly fluid model

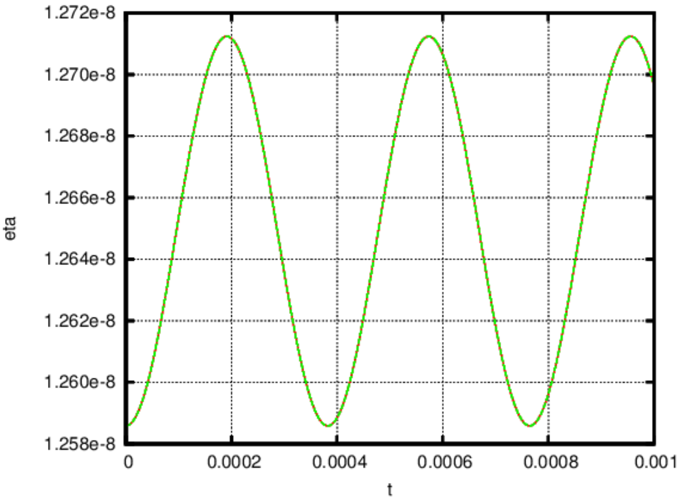


Figure: Exact solution for space-independent non-stationary η equation vs the numerical one

Conclusion

Resumé:

- Unconditionally hyperbolic bubbly fluid model is derived
- The numerical solution for the Riemann problem is built
- The comparison with the exact non-stationary solution is shown

Plans:

- Modulations equations for Green-Naghdi model
- 2D simulations

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