An asymptotic approach to the elastodynamic homogenization of periodic media

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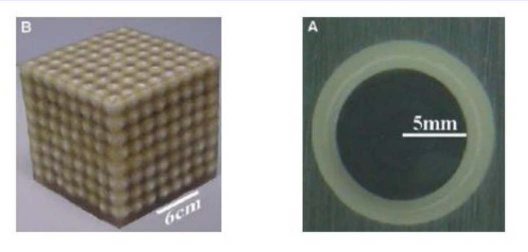
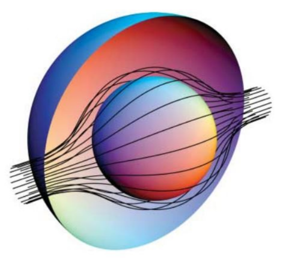
Background

Metamaterials

Metamaterials are artificial composites with special properties that cannot be found in nature.



Acoustic metamaterial (Liu et al. 2000[a])



Optical metamaterial (Pendry et al. 2006[b])

[a] Zhengyou Liu et al. "Locally resonant sonic materials". In: *science* 289.5485 (2000), pp. 1734–1736.

[b] John B Pendry, David Schurig, and David R Smith. "Controlling electromagnetic fields". In: *science* 312.5781 (2006), pp. 1780–1782.

Core developments of Willis' theory

- 1980s, two polarization fields were introduced for a fictional homogeneous comparison (Wills 1980a,b);
- The elastrodynamic homogenization theory of Willis was presented in Willis (1997);
- Asymptotic elastodynamic homogenization methods were proposed for periodic media (Bensoussan et al. 1978, Boutin and Auriault 1993, Craster et al. 2010[a]);
- Some extensions of Willis' theory have been proposed. (Milton and Willis 2007, Amirkhizi and Nemat-Nasser 2008[b], Nemat-Nasser et al. 2011, Nassar H. et al. 2015, etc.).

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[a] Richard V Craster, Julius Kaplunov, and Aleksey V Pichugin. "High-frequency homogenization for periodic media". In: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*. Vol. 466. 2120. The Royal Society. 2010, pp. 2341–2362.

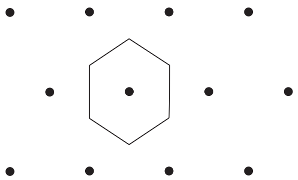
[b] Alireza V Amirkhizi and Sia Nemat-Nasser. "Microstructurally-based homogenization of electromagnetic properties of periodic media". In: *Comptes Rendus Mecanique* 336.1-2 (2008), pp. 24–33.

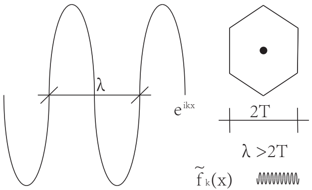
Preliminaries of homogenization theory

Periodic geometry

Consider a lattice $\mathscr{L}$ of the periodic vector space $\mathscr{E}$. The first Brillouin's zone T is defined : (same definitions for $\mathscr{E}^*$, $\mathscr{L}^*$ and T^*)

$$T = \{\boldsymbol{x} \in \mathscr{E} \mid \|\boldsymbol{x}\| < \|\boldsymbol{x} - \boldsymbol{y}\|, \boldsymbol{y} \in \mathscr{L} - \{\boldsymbol{0}\}\}$$





Floquet-Bloch transform

The Floquet-Bloch transform provides the definition

$$f(\boldsymbol{x}) = \int_{\boldsymbol{k} \in T} \tilde{f}_{\boldsymbol{k}}(\boldsymbol{x}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} d_{\boldsymbol{k}}$$

Motion equation

Using the FB transform for the constitutive relation and the momentum balance

$$(\boldsymbol{\nabla}+i\boldsymbol{k})\cdot\{\boldsymbol{C}(\boldsymbol{x}):[(\boldsymbol{\nabla}+i\boldsymbol{k})\otimes^{s}\tilde{\boldsymbol{u}}_{\boldsymbol{k}}(\boldsymbol{x})]\}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}+\tilde{\boldsymbol{f}}_{\boldsymbol{k}}(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}=-w^2\rho(\boldsymbol{x})\tilde{\boldsymbol{u}}_{\boldsymbol{k}}(\boldsymbol{x})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Effective field

The space average over the unit cell corresponds to the expected value of the wave amplitude at the local region :

$$\langle f(\boldsymbol{x})\rangle_{FB}=\frac{1}{|T|}\left(\int_{\boldsymbol{x}\in T}\tilde{f}_{\boldsymbol{k}}(\boldsymbol{x})d_{\boldsymbol{x}}\right)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\equiv\langle\tilde{f}\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Weighted average can be obtained via a random coefficient $\mathrm{w}(\boldsymbol{x},\alpha)$ with $\langle\mathrm{w}\rangle=1$ and α being the portion of each phase in the unit T (Milton and Willis 2007[a]) :

$$\boldsymbol{f}(\boldsymbol{x})\equiv\langle\mathrm{w}\tilde{f}(\boldsymbol{x})\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

[a] Graeme W Milton and John R Willis. "On modifications of Newton's second law and linear continuum elastodynamics". In: *Proceedings of the royal society of london A: Mathematical, Physical and Engineering Sciences*. Vol. 463. 2079. The Royal Society. 2007, pp. 855–880.

Homogenization theory

Localization step

The solution of the motion equation has a coupling relation with the effective strain and vector field (Willis 1997[a]).

$$\tilde{\boldsymbol{u}} = \langle \tilde{\boldsymbol{u}} \rangle + \boldsymbol{A} : \langle \tilde{\boldsymbol{\varepsilon}} \rangle + \boldsymbol{B} \cdot \langle \tilde{\boldsymbol{v}} \rangle$$

One approach to obtaining the two localization tensors is to introduce an eigen-strain field $\boldsymbol{\gamma}$ (FB wave expression are available).

$$(\boldsymbol{\nabla} + i\boldsymbol{k}) \cdot \{\boldsymbol{C} : [(\boldsymbol{\nabla} + i\boldsymbol{k}) \otimes^s \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{\gamma}}]\} + \tilde{\boldsymbol{f}} = -w^2 \rho \tilde{\boldsymbol{u}}$$

Green's function Introduce the Green's function $\boldsymbol{g}$ in order to make the homogenization motion equation solvable.

$$(\boldsymbol{\nabla} + i\boldsymbol{k}) \cdot \{\boldsymbol{C} : [(\boldsymbol{\nabla} + i\boldsymbol{k}) \otimes^s \boldsymbol{g}]\} + |T| \delta \boldsymbol{I} = -w^2 \rho \boldsymbol{g}$$

[a] John R Willis. "Dynamics of composites". In: *Continuum micromechanics*. Springer, 1997, pp. 265–290.

Homogenization step

The effective fields are simply defined as a volume average over the unit body T. The effective constitutive law is specified by

$$\begin{bmatrix} \boldsymbol{\Sigma} \\ \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}^e & \boldsymbol{S}^1 \\ \boldsymbol{S}^2 & \boldsymbol{\rho}^e \end{bmatrix}_{\boldsymbol{k},w} \begin{bmatrix} \boldsymbol{E} - \boldsymbol{\gamma} \\ \boldsymbol{V} \end{bmatrix}$$

where the tensor $\boldsymbol{S}^1$ and $\boldsymbol{S}^2$ are the third-order coupling tensors, which depend on the couple $(\boldsymbol{k}, w)$:

$$\boldsymbol{C}^e = \langle \boldsymbol{C} \rangle + \langle \boldsymbol{C} : [(\nabla_{\boldsymbol{y}} + i\boldsymbol{k}) \otimes^s \boldsymbol{A}] \rangle, \quad \boldsymbol{S}^2 = iw \langle \rho \boldsymbol{A} \rangle$$

$$\boldsymbol{S}^1 = \langle \boldsymbol{C} : [(\nabla_{\boldsymbol{y}} + i\boldsymbol{k}) \otimes^s \boldsymbol{B}] \rangle, \quad \boldsymbol{\rho}^e = \langle \rho \rangle \boldsymbol{I} + iw \langle \rho \boldsymbol{B} \rangle$$

The effective relation is independent of the prescribed initial condition.

Homogenization conditions

Virtual work condition The Hill-Mandel relation is still available in the dynamic case :

$$\int_T \tilde{\boldsymbol{f}}_{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}^*_{\boldsymbol{k}} d_{\boldsymbol{x}} = \int_T \tilde{\boldsymbol{F}}_{\boldsymbol{k}} \cdot \tilde{\boldsymbol{U}}^*_{\boldsymbol{k}}, \quad \boldsymbol{k} \in T$$

Effective field condition The "slow wave" wavelength λ is greater than the characteristic length of the unit cell length $2l$:

$$\lambda = |\frac{2\pi}{\boldsymbol{k}}| \geqslant 2l \quad \Rightarrow \quad |\boldsymbol{k}| \leqslant \frac{\pi}{l}$$

Energy condition Effective behavior is intended to describe the macroscopic properties of the composite :

$$\langle [\boldsymbol{C} : (\boldsymbol{\nabla} \otimes^s \tilde{\boldsymbol{u}})] : (\boldsymbol{\nabla} \otimes^s \tilde{\boldsymbol{u}}^*) \rangle \ll \langle [\boldsymbol{C} : (i\boldsymbol{k} \otimes^s \tilde{\boldsymbol{u}})] : (i\boldsymbol{k} \otimes^s \tilde{\boldsymbol{u}})^* \rangle$$

which presents a relation of an approximation condition (Nassar H. et al. 2015[a]) :

$$w^2 \lesssim max(\frac{c_I^m}{\rho^m}) \frac{\pi^2}{l_m^2}$$

[a] Hussein Nassar, Q-C He, and Nicolas Auffray. "Willis elastodynamic homogenization theory revisited for periodic media". In: *Journal of the Mechanics and Physics of Solids* 77 (2015), pp. 158–178.

Motion equations

Two-scale representation

It allows to research the macroscopic behaviour of a periodic medium at the microscopic scale (Bensoussan et al.1978[a]). Let us introduce macro-$\boldsymbol{x}$ and micro-$\boldsymbol{y}$ with

$$\boldsymbol{y} = \varepsilon^{-1}\boldsymbol{x}$$

The local motion equation expression :

$$(\boldsymbol{\nabla} + i\boldsymbol{k}) \cdot \{\boldsymbol{C}(\boldsymbol{y}) : [(\boldsymbol{\nabla} + i\boldsymbol{k}) \otimes^{s} \tilde{\boldsymbol{u}}(\boldsymbol{y})]\} + \tilde{\boldsymbol{f}} = -w^{2}\rho(\boldsymbol{y})\tilde{\boldsymbol{u}}(\boldsymbol{y})$$

The strain field expression :

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\boldsymbol{x}} + \frac{1}{\varepsilon}\boldsymbol{\varepsilon}_{\boldsymbol{y}} = \boldsymbol{\nabla}_{\boldsymbol{x}} \otimes^{s} \boldsymbol{u} + \frac{1}{\varepsilon}\boldsymbol{\nabla}_{\boldsymbol{y}} \otimes^{s} \boldsymbol{u}$$

[a] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou. "Asymptotic methods for periodic structures". In: *Stud. Math. Appl* 5 (1978).

Motion equations

Simplify the each order motion equation with the series expansion $\tilde{\boldsymbol{u}}^{\varepsilon} = \sum_r \varepsilon^r \tilde{\boldsymbol{u}}^r$, with $r \in N$:

$$
\begin{aligned}
\varepsilon^{-2}: \quad & \boldsymbol{\nabla}_{\boldsymbol{y}} \cdot [\boldsymbol{C} : (\boldsymbol{\nabla}_{\boldsymbol{y}} \otimes^s \tilde{\boldsymbol{u}}^0)] = \mathbf{0} \\
\varepsilon^{-1}: \quad & \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot [\boldsymbol{C} : (\boldsymbol{\nabla}_{\boldsymbol{y}} \otimes^s \tilde{\boldsymbol{u}}^0)] + \\
& \boldsymbol{\nabla}_{\boldsymbol{y}} \cdot [\boldsymbol{C} : (\boldsymbol{\nabla}_{\boldsymbol{x}} \otimes^s \tilde{\boldsymbol{u}}^0 + \boldsymbol{\nabla}_{\boldsymbol{y}} \otimes^s \tilde{\boldsymbol{u}}^1)] = \mathbf{0} \\
\varepsilon^{0}: \quad & \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot [\boldsymbol{C} : (\boldsymbol{\nabla}_{\boldsymbol{x}} \otimes^s \tilde{\boldsymbol{u}}^0 + \boldsymbol{\nabla}_{\boldsymbol{y}} \otimes^s \tilde{\boldsymbol{u}}^1)] + \\
& \boldsymbol{\nabla}_{\boldsymbol{y}} \cdot [\boldsymbol{C} : (\boldsymbol{\nabla}_{\boldsymbol{x}} \otimes^s \tilde{\boldsymbol{u}}^1 + \boldsymbol{\nabla}_{\boldsymbol{y}} \otimes^s \tilde{\boldsymbol{u}}^2)] + \boldsymbol{f} = -w^2 \rho \boldsymbol{u}_0 \\
& \ldots\ldots \\
\varepsilon^{n-1}: \quad & \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot [\boldsymbol{C} : (\boldsymbol{\nabla}_{\boldsymbol{x}} \otimes^s \tilde{\boldsymbol{u}}^{n-1} + \boldsymbol{\nabla}_{\boldsymbol{y}} \otimes^s \tilde{\boldsymbol{u}}^n)] + \\
& \boldsymbol{\nabla}_{\boldsymbol{y}} \cdot [\boldsymbol{C} : (\boldsymbol{\nabla}_{\boldsymbol{x}} \otimes^s \tilde{\boldsymbol{u}}^n + \boldsymbol{\nabla}_{\boldsymbol{y}} \otimes^s \tilde{\boldsymbol{u}}^{n+1})] = -w^2 \rho \boldsymbol{u}_{n-1} \quad n \in N^*
\end{aligned}
$$

Solutions

Comparing the orders of the parameter ε, we get the solution of the first four equations :

$$\begin{aligned}
\boldsymbol{u}_0 &= \tilde{U}_0 \\
\boldsymbol{u}_1 &= \tilde{U}_1 + \mathscr{X}_1 \boldsymbol{\nabla}_{\boldsymbol{x}} \tilde{U}_0 \\
\boldsymbol{u}_2 &= \tilde{U}_2 + \mathscr{X}_1 \boldsymbol{\nabla}_{\boldsymbol{x}} \tilde{U}_1 + \mathscr{X}_2 \boldsymbol{\nabla}^2_{\boldsymbol{x}} \tilde{U}_0 + \mathscr{H}_2 \tilde{\boldsymbol{f}} \\
\boldsymbol{u}_3 &= \tilde{U}_3 + \mathscr{X}_1 \boldsymbol{\nabla}_{\boldsymbol{x}} \tilde{U}_2 + \mathscr{X}_2 \boldsymbol{\nabla}^2_{\boldsymbol{x}} \tilde{U}_1 + \mathscr{X}_3 \boldsymbol{\nabla}^3_{\boldsymbol{x}} \tilde{U}_0 + \mathscr{H}_3 \boldsymbol{\nabla}_{\boldsymbol{x}} \tilde{\boldsymbol{f}}
\end{aligned}$$

where the series matrices $\mathscr{H}_i$ are derived from the density difference of composite materials :

$$e.g. \quad \boldsymbol{\nabla}_{\boldsymbol{y}} \cdot [\boldsymbol{C} : \boldsymbol{\nabla}_{\boldsymbol{y}} \mathscr{H}_2(\boldsymbol{y})] \tilde{\boldsymbol{f}} = (\rho \langle \rho \rangle^{-1} - \boldsymbol{I}) \tilde{\boldsymbol{f}}$$

Assumption

As mentioned earlier, the body force and external volume loading have a large impact on the effective impedance expressions. Therefore, in the absence of body force, the effective impedance can be simplified so as to reduce to a regular solution equivalent to the work of Boutin and Auriault (1993[a]) :

$$\tilde{\boldsymbol{u}} = \sum_{i=0}^{n} \varepsilon^i \mathscr{X}_i \boldsymbol{\nabla}_{\boldsymbol{x}}^i \Big(\sum_{i=0}^{n} \varepsilon^i \tilde{U}_i\Big) + O(\varepsilon^{n+1}), \quad \mathscr{X}_0 = \boldsymbol{I},\ \boldsymbol{\nabla}_{\boldsymbol{x}}^0 = \boldsymbol{I}$$

Therefore, the average displacement field takes the form

$$\langle \tilde{\boldsymbol{u}} \rangle = \sum_{i=0}^{n} \varepsilon^i \tilde{U}_i$$

[a] C Boutin and JL Auriault. "Rayleigh scattering in elastic composite materials". In: *International journal of engineering science* 31.12 (1993), pp. 1669–1689.

Effective impedance

Hierarchical motion equation

Using the displacement expansion $\boldsymbol{u}^{\varepsilon} = \sum_n \varepsilon^n \boldsymbol{u}_n$ with $n \in N$,

$$\begin{aligned}
\varepsilon^0: &\quad \bar{\boldsymbol{Z}}^0 \tilde{U}_0 = \tilde{\boldsymbol{f}} \\
\varepsilon^1: &\quad \bar{\boldsymbol{Z}}^0 \tilde{U}_1 + \bar{\boldsymbol{Z}}^1 \tilde{U}_0 = \tilde{\boldsymbol{Z}}^1 \tilde{\boldsymbol{f}} \\
\varepsilon^2: &\quad \bar{\boldsymbol{Z}}^0 \tilde{U}_2 + \bar{\boldsymbol{Z}}^1 \tilde{U}_1 + \bar{\boldsymbol{Z}}^2 \tilde{U}_0 = \tilde{\boldsymbol{Z}}^2 \tilde{\boldsymbol{f}}
\end{aligned}$$

Ignoring the higher order small items,

$$\boldsymbol{Z}^2 = (\boldsymbol{I} + \varepsilon \tilde{\boldsymbol{Z}}^1 + \varepsilon^2 \tilde{\boldsymbol{Z}}^2)^{-1} (\bar{\boldsymbol{Z}}^0 + \varepsilon \bar{\boldsymbol{Z}}^1 + \varepsilon^2 \bar{\boldsymbol{Z}}^2)$$

with

$$\begin{aligned}
\bar{\boldsymbol{Z}}^n &= iw\langle \rho \mathscr{X}_n \rangle iw - i\boldsymbol{k} \langle \boldsymbol{C}(\mathscr{X}_n + \nabla_{\boldsymbol{y}} \mathscr{X}_{n+1}) \rangle i\boldsymbol{k} \cdot (i\boldsymbol{k})^n, \; (n = 0, 1, 2, \; with \; \mathscr{X}_0 = \boldsymbol{I}) \\
\tilde{\boldsymbol{Z}}^n &= i\boldsymbol{k} \langle \boldsymbol{C} : (\nabla_{\boldsymbol{y}} \mathscr{H}_{n+1} + \mathscr{H}_n) \rangle i\boldsymbol{k} + iw \langle \rho \mathscr{H}_n \rangle, \; (n = 1, 2, \; with \; \mathscr{H}_1 = \boldsymbol{0})
\end{aligned}$$

Displacement series expression

Set the lowest order expression like $\boldsymbol{u}^{\varepsilon} = \boldsymbol{u}_0 + O(\varepsilon)$, the lowest order motion equation has been defined by :

$$\boldsymbol{Z}^0 = \boldsymbol{k} \cdot \langle \boldsymbol{C} \rangle \cdot \boldsymbol{k} - w^2 \langle \rho \rangle \boldsymbol{I}$$

With the same way, set $\boldsymbol{u}^{\varepsilon} = \boldsymbol{u}_0 + \varepsilon \boldsymbol{u}_1 + \varepsilon^2 \boldsymbol{u}_2 + O(\varepsilon^3)$,

$$\begin{aligned}
\boldsymbol{Z}^2 &\in \{\boldsymbol{\gamma} \boldsymbol{Z}^2_{min}, \boldsymbol{\gamma} \boldsymbol{Z}^2_{max}\} \\
\boldsymbol{Z}^2_{min} &= \tilde{\boldsymbol{Z}}^0 + \varepsilon \tilde{\boldsymbol{Z}}^1 + \varepsilon^2 \tilde{\boldsymbol{Z}}^2 - \hat{\boldsymbol{Z}}^1 - \varepsilon \hat{\boldsymbol{Z}}^2 \\
\boldsymbol{Z}^2_{max} &= \tilde{\boldsymbol{Z}}^0 + \varepsilon \tilde{\boldsymbol{Z}}^1 + \varepsilon^2 \tilde{\boldsymbol{Z}}^2 - \hat{\boldsymbol{Z}}^1
\end{aligned}$$

With,

$$\begin{aligned}
\tilde{\boldsymbol{Z}}^i &= iw \langle \rho \mathscr{X}_i \rangle iw - i\boldsymbol{k} \langle \boldsymbol{C} : \mathscr{X}_i \rangle i\boldsymbol{k} \quad with \quad \mathscr{X}_0 = \boldsymbol{I} \\
\hat{\boldsymbol{Z}}^i &= i\boldsymbol{k} \langle \boldsymbol{C} : \nabla_{\boldsymbol{y}} \mathscr{X}_i \rangle i\boldsymbol{k} \quad with \quad \mathscr{X}_0 = \boldsymbol{I} \\
\boldsymbol{\gamma} &= I + \varepsilon i\boldsymbol{k} \langle \boldsymbol{C} : \nabla_{\boldsymbol{y}} \mathscr{H}_2 \rangle + \varepsilon^2 (i\boldsymbol{k} \langle \boldsymbol{C} : \mathscr{H}_2 \rangle i\boldsymbol{k} - iw \langle \rho \mathscr{H}_2 \rangle)
\end{aligned}$$

Dispersion relation

The motion equation for a simple two phase periodic structure :

$$E_n \frac{(u_{n+1} - u_n)}{a} - E_{n-1} \frac{(u_n - u_{n-1})}{a} + f_n = -w^2 a \rho_n u_n$$

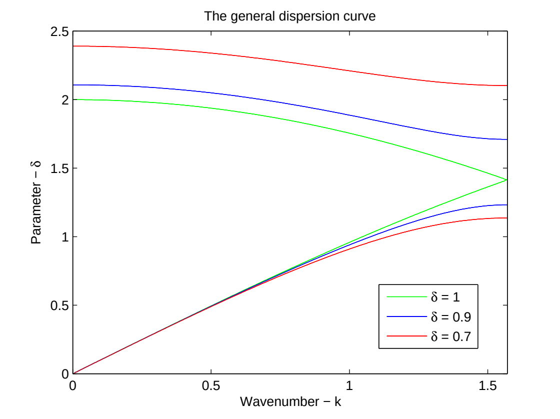
According for the periodic boundary conditions

$$\delta \upsilon^4/4 - \upsilon^2 + sin^2(ak) = 0$$

with

$$w_0^2 = \frac{\langle E \rangle}{\langle \rho \rangle} = \frac{4E_1E_2}{a(\rho_1+\rho_2)(E_1+E_2)}, \quad w_i^2 = \frac{E_i}{a\rho_i} \ (with\ i = 1,2)$$

$$\delta = \frac{16E_1E_2\rho_1\rho_2}{(E_1+E_2)^2(\rho_1+\rho_2)^2} = (\frac{w_0}{w_1})^2(\frac{w_0}{w_2})^2, \quad \upsilon^2 = \frac{(E_1+E_2)(\rho_1+\rho_2)}{4E_1E_2}(aw)^2 = (\frac{w}{w_0})^2$$



The width of the “bandgap” is largely influenced by the structure of the composite :

$$\upsilon \in \left[0, \sqrt{\frac{2-2\sqrt{1-\delta sin^2(ak)}}{\delta}}\right] \bigcup \left[\sqrt{\frac{2+2\sqrt{1-\delta sin^2(ak)}}{\delta}}, \frac{2}{\sqrt{\delta}}\right], \quad \forall |k| \leqslant \frac{\pi}{2a}$$

Analysed solution

Rewrite the motion equation in the matrix form.

$$\{[\tilde{\boldsymbol{\nabla}}]^T[\boldsymbol{C}][\tilde{\boldsymbol{\nabla}}]-w^2[\rho]\}[\tilde{u}]=[\mathscr{K}][\tilde{u}]=\tilde{\boldsymbol{f}}$$

Combine the continuity and periodic conditions for the displacement $\boldsymbol{u}$ and stress $\boldsymbol{\sigma}$, and note that $[\mathscr{P}][\tilde{u}]=0$:

$$\left[\mathscr{K}+\mathscr{P}\right]\left[\tilde{u}\right]=\left[\tilde{f}\right]$$

The dispersion relation is defined by " $det\{\left[\mathscr{K}+\mathscr{P}\right]\}=0$ ".
Get the dispersion relation as in the work of Nassar H. et al. (2016[a]) :

$$cos(2ka)=\frac{(\sqrt{\rho_1E_1}+\sqrt{\rho_2E_2})^2}{4\sqrt{\rho_1E_1\rho_2E_2}}cos((\sqrt{\frac{\rho_1}{E_1}}+\sqrt{\frac{\rho_2}{E_2}})wa)-\frac{(\sqrt{\rho_1E_1}-\sqrt{\rho_2E_2})^2}{4\sqrt{\rho_1E_1\rho_2E_2}}cos((\sqrt{\frac{\rho_1}{E_1}}-\sqrt{\frac{\rho_2}{E_2}})wa)$$

[a] Hussein Nassar, Q-C He, and Nicolas Auffray. "On asymptotic elastodynamic homogenization approaches for periodic media". In: *Journal of the Mechanics and Physics of Solids* 88 (2016), pp. 274–290.

FEM solution

The weak form of the integral equation :

$$\int_T \boldsymbol{C} : \tilde{\boldsymbol{\varepsilon}}(\tilde{\boldsymbol{u}}) : \tilde{\boldsymbol{\varepsilon}}(\delta\tilde{\boldsymbol{u}})^* d_T - w^2 \int_T \rho \tilde{\boldsymbol{u}} \cdot \delta\tilde{\boldsymbol{u}}^* d_T = \int_T \tilde{\boldsymbol{f}} \cdot \delta\tilde{\boldsymbol{u}}^* d_T$$

Set $\mathscr{Z}_b U_b = 0$ to represent all the periodic boundary conditions. The global motion equation is defined as :

$$\left(\begin{bmatrix} [K] & 0 \\ 0 & \mathscr{Z}_b \end{bmatrix} - w^2 \begin{bmatrix} [M] & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} U \\ U_b \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

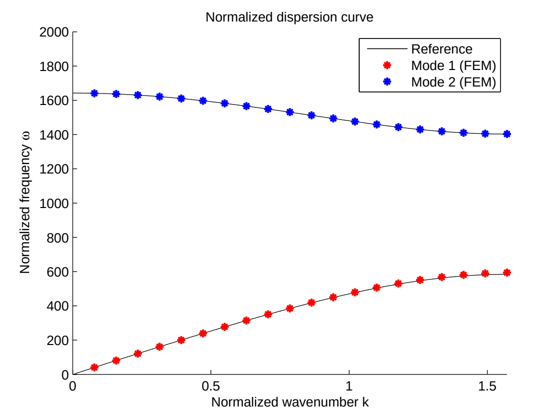
The effective impedance has the following dispersion relation :

$$[K]_{glob} \boldsymbol{V} = \lambda [M]_{glob} \boldsymbol{V}$$

where the generalized eigenvalue λ represents the square of the angular frequency w^2 and $\boldsymbol{V}$ stands for the generalized eigenvectors.

Two-layer example

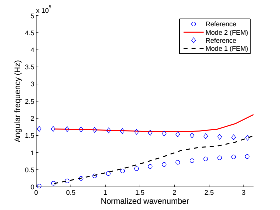
The results of the finite element simulation and the analytical solution are compared.



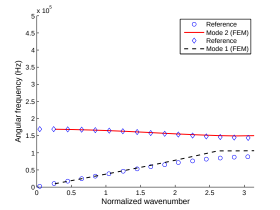
The Young's modulus (Pa), $E_1 = 3.0e^8, E_2 = 2.0e^{11}$
The density (kg/m^3): $\rho_1 = 1.5e^3, \rho_2 = 3.0e^3$
The unit characteristic size $l = 5.0e^{-3}(m)$

Multi-layer example

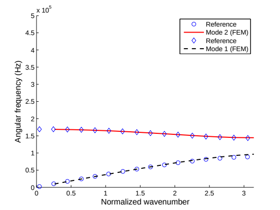
Comparison with the results by Nemat-Nasser et Srivastava (2011[a]).



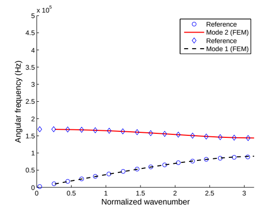
$\delta x = 0.0001m$



$\delta x = 0.00005m$



$\delta x = 0.00002m$



$\delta x = 0.00001m$

[a] Sia Nemat-Nasser et al. "Homogenization of periodic elastic composites and locally resonant sonic materials". In: *Physical Review B* 83.10 (2011), p. 104103.

Conclusion

Conclusions

- The influence of body force term on the asymptotic effective impedance has been discussed;
- The FEM results has been compared with the analytical results and analysed the influence of mesh size on the numeric result;
- Two higher order asymptotic expressions of the effective impedances have been derived.

Works to be done

- Verify the validity of the high-order effective impedance expressions.
- Study the dynamic homogenization of the motion equation involving the non-uniformly body force function.