An hyperbolic model of nonlinear acoustics with Helmholtz resonators

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Sommaire

Solitons

- solitons : nonlinear waves, large amplitude and constant profile
  ✓ competition between nonlinearity and dispersion
  ✓ many physical systems : fluid dynamics (KdV), optics, ...
- acoustics : no solitons
  ✗ intrinsic dispersion too low
  ☞ need to introduce geometric dispersion

Acoustic solitons

- tube with Helmholtz resonators
  - ✓ Sugimoto (📚 JFM 92, 04) : physical modeling
  - ✓ Richoux, Lombard, Mercier (📚 Wave Motion 15) : numerics, experiments

O. Richoux, LAUM

Sugimoto's model

Hypotheses

- low-frequency : propagating plane mode $\rightarrow$ 1D
- weak acoustic nonlinearity in the tube
- $\lambda \gg$ distance $\rightarrow$ continuous distribution of resonators

Evolution equations

- field splitted into simple right-going (+) and left-going waves (-)

$$\begin{cases} \dfrac{\partial u^\pm}{\partial t} + \dfrac{\partial}{\partial x}\left(\pm a u^\pm + b\dfrac{(u^\pm)^2}{2}\right) = \pm c \dfrac{\partial^{-1/2}}{\partial t^{-1/2}}\dfrac{\partial u^\pm}{\partial x} + d\dfrac{\partial^2 u^\pm}{\partial x^2} \mp e\dfrac{\partial p^\pm}{\partial t} \\ \dfrac{\partial^2 p^\pm}{\partial t^2} + f\dfrac{\partial^{3/2} p^\pm}{\partial t^{3/2}} + g p^\pm - m\dfrac{\partial^2 (p^\pm)^2}{\partial t^2} + n\left|\dfrac{\partial p^\pm}{\partial t}\right|\dfrac{\partial p^\pm}{\partial t} = \pm h u^\pm \end{cases}$$

▷ propagation : linear a and nonlinear b

▷ oscillations : linear g and nonlinear m

▷ coupling : e, h

▷ attenuation : d, fractional c and f, nonlinear n

A simplified model

Hypothesis : right-going waves,
no fractional attenuation, no nonlinear losses,

$$\begin{cases} \partial_t u \quad + a\partial_x u + b\partial_x \left(\frac{u^2}{2}\right) & = & -\Omega^2 \varphi, \\ \partial_t \varphi & = & u - \varepsilon\varphi - \omega_0^2 \phi, \\ \partial_t \phi & = & \varphi, \end{cases}$$

$\Omega > 0$, $\omega_0 > 0$, $a > 0$, $b > 0$, $\varepsilon \geq 0$, with initial data,

$$u(0,x) = u_0(x), \qquad \varphi(0,x) = \varphi_0(x), \qquad \phi(0,x) = \phi_0(x),$$

Non increasing energy for smooth solution

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left(u^2 + \Omega^2\varphi^2 + \Omega^2\omega_0^2\phi^2\right)(x,t)\,dx = -\Omega^2\varepsilon \int \varphi^2 d\,x.$$

Rewritten system as an equation

Burgers-Helmholtz system

$$\partial_t u + a\partial_x u + b\partial_x \left(\frac{u^2}{2}\right) = -\Omega^2 \varphi,$$
$$\partial_t^2 \varphi + \varepsilon \partial_t \varphi + \omega_0^2 \varphi = \partial_t u$$

as a Burgers equation with a nonlocal source term

$$\partial_t u + a\partial_x u + b\partial_x \left(\frac{u^2}{2}\right) = -u_0(x) K_1(t) + \int_0^t K(t-s) u(s,x) ds.$$

Basics on the Burgers equation

$$\partial_t u + a\partial_x u + b\partial_x \left(\frac{u^2}{2}\right) = 0, \qquad u(0,x) = u_0(x)$$

1. Maximum principle : $\inf u_0(x) \leq u(t,x) \leq \sup u_0(x)$
2. Global smooth solution if and only if $u_0 \uparrow$
3. apparition of a shock wave in a finite time
4. **Unique global weak entropy solution** $\forall u_0 \in L^\infty$: Kruzkov 1970
5. Smoothing effect $\forall t > 0,\ u(t,.) \in BV_x(\mathbb{R},\mathbb{R})$: Lax & Oleinik 1957

Existence of some small smooth solutions

The system is *partially* dissipative.

The Kawashima condition (K) (1985) ensures the existence of global (small) smooth solutions near equilibrium.

(K) means for the linearized system $\partial_t U + A\partial_x U = SU$ at some equilibrium that there is no eigenvectors of the linearized flux in the kernel of the linearized source.

$$AV = \lambda V \quad \& \quad V \neq \vec{0} \quad \Rightarrow \quad S\,V \neq \vec{0}$$

If (K) is not satisfied some travelling waves are not dissipated
If $AV = \lambda V$ and $SV = 0$ then the travelling wave $U(t, x) = v(x - \lambda t)V$ where v is a scalar function is not dissipated by the source term.

Global smooth solutions around non zero equilibrium

Proposition ((K)=(Kawashima) condition)

The (K) condition is fulfilled at all non zero equilibrium wich is parametrized by $\phi_e \neq 0$,

$$(u_e, \varphi_e, \phi_e) = (\omega_0^2 \phi_e, 0, \phi_e);$$

but, the (K) condition is not satisfied at rest

$$(u_e, \varphi_e, \phi_e) = (0, 0, 0).$$

and global smooth solutions are expected (?)

Burgers equation with a dissipative source term

With a dissipative source term more smooth solutions than without.
The simplest example

$$\partial_t u + u\partial_x u = -L\, u$$

- global smooth solution $\Longleftrightarrow -L \leq u_0'(x)$.
- Lax proof : $v = \partial_x u$
  $(\partial_t + u\partial_x)v = -v^2 - Lv = -v(v+L)$
  $-\infty \quad \leftarrow \quad -L \quad \rightarrow \quad 0 \quad \leftarrow \quad +\infty$
  $v = \partial_x u = 0$ attractive equilibrium for all $-L < u$
  $\partial_x u = -\infty$ attractive "blow-up" (Lax-shock) for all $u < -L$

1. Method generalized by Lax for 2×2 system then ... Majda ...
2. Burgers-Hilbert, Bressan-Khai, SIAM Math. Anal. 2014.

Theorem (Shock in finite time)

Apparition of schok wave for smooth bounded initial data with

$$\inf \partial_x u_0(x) \ll 0 \leq \sup \partial_x u_0(x) \ll |\inf \partial_x u_0(x)|$$

A "Lax" proof : $v = \partial_x u$

$$\frac{d\,v}{d\,t} + b\,v^2 = k(t,x) - \mathcal{L}(v).$$

simplified dynamics along characteristics :

$$\frac{d\,v}{d\,t}(x,t) \leq -b\,v^2(x,t) + C_0 + C_\varepsilon \sup_{(y,s)\in\mathbb{R}\times[0,t]} |v(y,s)|.$$

Global weak entropy solutions

Rewrite an equation as a system
Classic idea : $v(x)$ not smooth, Baiti-Jenssen J.D.E. 1997

$$\partial_t u + \partial_x f(v(x), u) = 0 \Longleftrightarrow \begin{cases} \partial_t u + \partial_x f(v, u) = 0 \\ \partial_t v = 0 \end{cases}$$

$N \times N$ system : $u = u_1$, $U = (u, u_2, \ldots, u_N)$,

$$\begin{cases} \partial_t u + \partial_x f(u) & = & \displaystyle\sum_{j=1}^{N} a_{1j} u_j, \\ \partial_t u_i & = & \displaystyle\sum_{j=1}^{N} a_{ij} u_j, \quad 1 < i \leq N \end{cases}$$

Existence via a splitting scheme

1. Solving on $[t, t+\Delta t[$

$$\partial_t u + \partial_x f(u) = 0$$

2. Next, again on $[t, t+\Delta t[$ with the previous final data at $t+\Delta t$ as the initial data of $u = u_1$ at t for the ODEs

$$\partial_t U = AU$$

$U = U(t, x)$.

In general NO "Maximum principle" : No Existence of invariant region

Entropy solutions

U is an entropy solution of the $N \times N$ system if $\forall \eta$ convex, $q' = \eta' f'$,

$$\forall \eta \geq 0, \quad \partial_t \eta(u) + \partial_x q(u) \leq \eta'(u) \sum_{j=1}^{N} a_{1j} u_j \quad \in \mathcal{D}'$$

Theorem (Existence and uniqueness of entropy solution)

$\forall T > 0, \exists !$ *entropy solution of the $N \times N$ system*
$U \in L^\infty([0,T] \times \mathbb{R}, \mathbb{R}^N) \cap C^0([0,T], L^1(\mathbb{R}, \mathbb{R}^N))$
Moreover if the initial data $U_0(x) \in BV^s$, $0 < s \leq 1$, then
$U \in L^\infty([0,T], BV^s(\mathbb{R}, \mathbb{R}^N)) \cap C^s([0,T], L^{1/s}(\mathbb{R}, \mathbb{R}^N))$.

Existence and uniqueness for Burgers-Helmholtz

- The same existence and uniqueness result

  So the Lax entropy condition on the velocity $[u] \leq 0$
  and no condition on $[\varphi]$ and $[\phi]$
- with a decreasing energy due to shock waves :

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \left(u^2 + \Omega^2\varphi^2 + \Omega^2\omega_0^2\phi^2\right)(x,t)\,dx \quad \leq \quad -\Omega^2\varepsilon \int \varphi^2 d\,x.$$

  The energy is a "mathematical" entropy

- No “Maximum principle”

Conclusions & Prospects

1. A simplified system built on Burgers equation for Burgers-Helmholtz resonators
2. Smooth solutions, Shock Waves, unique entropy solutions

3. More global smooth solutions are expected ?
4. Kruzkov 1970 : scalar conservation laws, $x = (x_1, \dots, x_d)$

$$\partial_t u + \mathrm{div} F(t, x, u) = g(t, x, u)$$

   Generalisation with variables coefficients ?
   for the multi-dimensional case ?
5. With fractional time derivatives ?
   via the hyperbolic approximated system ?
   Bruno Lombard & Denis Matignon, SIAM Appl. Math. 2016