Fluids containing small gas bubbles and Co

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GDR MecaWave, Marseille, 5 October, 2018

General framework

Classical Lagrangian :

$$\mathcal{L} = \int_{\Omega(t)} \rho \left(\frac{|\mathbf{u}|^2}{2} - e\left(\mathbf{F}\right) \right) d\Omega, \quad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}.$$

Generalized Lagrangian :

$$\mathcal{L} = \int_{\Omega(t)} \rho \left(\frac{|\mathbf{u}|^2}{2} - \tilde{e}\left(\mathbf{F}, \boldsymbol{\nabla}\mathbf{F}, \dot{\mathbf{F}}\right) \right) d\Omega.$$

The simplest case :

$$\mathcal{L} = \int_{\Omega(t)} \rho \left(\frac{|\mathbf{u}|^2}{2} - \tilde{e}\left(\rho, \boldsymbol{\nabla}\rho, \dot{\rho}\right) \right) d\Omega.$$

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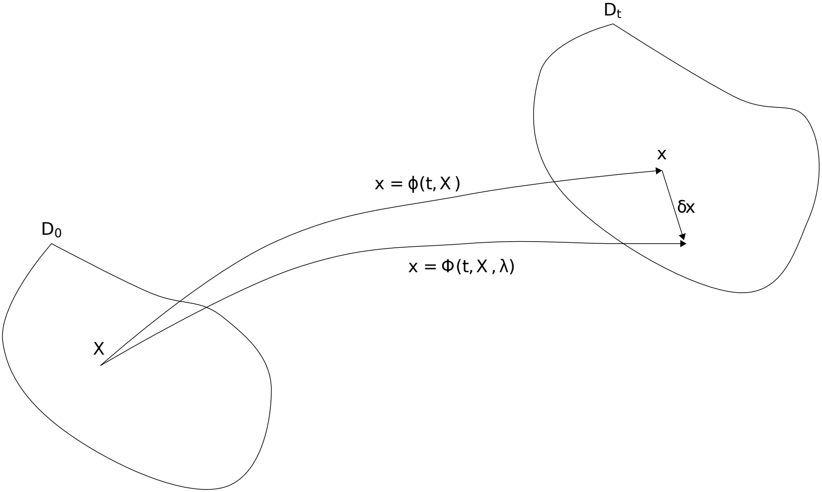
Motion and virtual motion



Motion and virtual motion

- $\mathbf{x} = \varphi(t, \mathbf{X}),\ \mathbf{x} = \left(x^1, x^2, x^3\right)^T,\ \mathbf{X} = \left(X^1, X^2, X^3\right)^T,$
- $\mathbf{x} = \boldsymbol{\Phi}(t, \mathbf{X}, \lambda),$
- $\delta\mathbf{x}(t, \mathbf{X}) = \frac{\partial}{\partial\lambda}\boldsymbol{\Phi}(t, \mathbf{X}, \lambda)|_{\lambda=0},$
- $\zeta(t, \mathbf{x}) = \delta\mathbf{x}\left(t, \varphi^{-1}(t, \mathbf{x})\right).$

Lagrangian variations

- $\tilde{\delta}\rho = -\rho div(\zeta)$,
- $\tilde{\delta}\mathbf{u} = \frac{\partial \delta \mathbf{x}}{\partial t}$.

Eulerian variations

- $\hat{\delta}\rho = -div(\rho\boldsymbol{\zeta})$,
- $\hat{\delta}\mathbf{u} = \frac{D\zeta}{Dt} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\zeta$.

Capillarity and inertia

Gradient ("Capillarity") type dispersion (D. Korteweg, J. D. van der Waals, M. Eglit, P. Casal, M. Slemrod, H. Gouin, L. Truskinovsky, P. Germain, S. Lurie, E. Aifantis, ...)

$$\mathcal{L} = \int_{\Omega(t)} \rho \left(\frac{|\mathbf{u}|^2}{2} - e\,(\rho, \boldsymbol{\nabla}\rho) \right) d\Omega.$$

"Inertia " type dispersion (S. V. Iordansky, B. Kogarko, L. van Wijngaarden, SG and Teshukov, ...)

$$\mathcal{L} = \int_{\Omega(t)} \rho \left(\frac{|\mathbf{u}|^2}{2} - \tilde{e}\,(\rho, \dot{\rho}) \right) d\Omega.$$

Capillarity and quantum mechanics

$$i\psi_t + \frac{1}{2}\Delta\psi - |\psi|^2\,\psi = 0, \tag{1}$$

where Δ is the Laplacian with respect to the space coordinates $\mathbf{x}$.
Equivalent 'hydrodynamic' form by Madelung's transformation :

$$\psi(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)}\,e^{i\theta(\mathbf{x},t)}. \tag{2}$$

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 \\ \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla\left(f(\rho) - \dfrac{\Delta(\sqrt{\rho})}{2\sqrt{\rho}}\right) = 0, \ \mathbf{u} = \nabla\theta. \end{cases} \tag{3}$$

Lagrangian for cubic NLS equation

$$\mathcal{L} = \int_{\Omega_t} \left(\rho \frac{|\mathbf{u}|^2}{2} - \rho e(\rho, \boldsymbol{\nabla}\rho) \right) d\Omega, \ \rho e(\rho, \boldsymbol{\nabla}\rho) = \frac{\rho^2}{2} + \frac{1}{4\rho} \frac{|\nabla\rho|^2}{2}. \quad (4)$$

Equations admit the energy conservation law of the form :

$$\frac{\partial E}{\partial t} + \operatorname{div}\left(E\mathbf{u} + \boldsymbol{\Pi}\mathbf{u} - \frac{\dot{\rho}}{4\rho}\nabla\rho \right) = 0, \quad (5)$$

where

$$E = \rho \frac{|\mathbf{u}|^2}{2} + \rho e(\rho, \boldsymbol{\nabla}\rho). \quad (6)$$

Lagrangian for fluids with inertia

$$\mathcal{L} = \int_{\Omega_t} \left(\rho \frac{|\mathbf{u}|^2}{2} - W(\rho, \dot{\rho}) \right) d\Omega, \;\; W(\rho, \dot{\rho}) = \rho \tilde{e}(\rho, \dot{\rho}).$$

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \mathrm{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + p\mathbf{I}\right) = 0, \\ p = \rho \dfrac{\delta W}{\delta \rho} - W, \end{cases} \tag{7}$$

Fluids with inertia : 1D case

Mass Lagrangian coordinate :

$$q = \int_0^X \rho_0(s)ds.$$

Lagrangian

$$\mathcal{L} = \int_{q_0}^{q_1} \left(\frac{u^2}{2} - \tilde{e}(\tau, \tau_t) \right) dq, \ \tau = \frac{1}{\rho}.$$

Equations

$$\tau_t - u_q = 0, \ u_t + p_q = 0, \ p = -\frac{\delta \tilde{e}}{\delta \tau}.$$

The general system admits the energy conservation law :

$$\left(e + \frac{1}{2}u^2 \right)_t + (pu)_q = 0, \ e = \tilde{e} - \frac{\partial \tilde{e}}{\partial \tau_t}\tau_t.$$

1D solid mechanics

Governing equations

$$w_t - u_q = 0, \quad u_t - \sigma_q = 0, \tag{8}$$

Here w is the longitudinal strain, u is the velocity, σ is the stress :

$$\sigma = \frac{\delta \tilde{e}}{\delta w} = \frac{\partial \tilde{e}}{\partial w} - \frac{\partial}{\partial t}\left(\frac{\partial \tilde{e}}{\partial w_t}\right) \tag{9}$$

The energy conservation law :

$$\left(e + \frac{1}{2}u^2\right)_t - (\sigma u)_q = 0, \; e = \tilde{e} - \frac{\partial \tilde{e}}{\partial w_t} w_t.$$

Circular rod with the lateral inertia effects

For a circular rod of radius a the effects of lateral inertia result to the potential (Rayleigh) :

$$\tilde{e}(w, w_t) = \frac{E}{2\rho_0} w^2 - \frac{\nu^2 a^2}{4} w_t^2,$$

where ρ_0 is the material density, E is Young's modulus, and ν is Poisson's ratio.

Fluid mechanics : Serre-Su-Gardner-Green-Naghdi equations

- Equations in Eulerian coordinates

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0,$$

$$\frac{\partial hu}{\partial t} + \frac{\partial hu^2 + p}{\partial x} = 0.$$

h- fluid depth

u - average velocity

$$p = \frac{gh^2}{2} + \frac{1}{3}h^2\ddot{h}$$

$$\dot{h} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x},$$

$$\ddot{h} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\dot{h}.$$

Lagrangian (Salmon 1988, SG & Teshukov 2001)

$$\mathcal{L} = \int_{-\infty}^{\infty} \left(\frac{hu^2}{2} - W(h, \dot{h}) \right) dx,$$

$$W(h, \dot{h}) = h\tilde{e}(\tau, \dot{\tau}), \;\; \tilde{e}(\tau, \dot{\tau}) = \frac{g}{2\tau} - \frac{\dot{\tau}^2}{6\tau^4}.$$

Multi-phase flows : “bubbly” fluids

Applications

1. Protecting underwater structures by “bubble curtains”.
2. Study of mechanical properties of irradiated metals with helium bubbles inside.
3. ...

Multi-phase flows : dilute one-velocity bubbly fluid

- Bubbles are spherical and have locally the same average radius R
- Bubbles are compressible
- Distance between bubbles is large (dilute mixture)
- Sliding between the surrounding fluid and bubbles is absent
- The surrounding fluid is incompressible $\rho_1 = \rho_{10} = const$.
- No phase transition
- No bubble breakup (the number of bubbles per unit volume N is conserved)

Volume fraction of bubbles :

$$\alpha_2 = \frac{4}{3}\pi R^3 N, \ \alpha_1 = 1 - \alpha_2.$$

Average density

$$\rho = \alpha_1 \rho_{10} + \alpha_2 \rho_2$$

Constraints

$$(\alpha_2 \rho_2)_t + div(\alpha_2 \rho_2 \mathbf{u}) = 0, \ \rho_t + div(\rho \mathbf{u}) = 0, \ N_t + div(N\mathbf{u}) = 0.$$

Lagrangian of the mixture

Total energy

$$E = T + W = \frac{\rho|\mathbf{u}|^2}{2} + 2\pi\rho_{10}R^3\dot{R}^2N + \alpha_2\rho_2\varepsilon_2(\rho_2).$$

Local Lagrangian :

$$L = T - W = \frac{\rho|\mathbf{u}|^2}{2} + 2\pi\rho_{10}R^3\dot{R}^2N - \alpha_2\rho_2\varepsilon_2(\rho_2).$$

Multi-phase flows : dilute one-velocity bubbly fluid

$$L = L(\rho, \dot{\rho}, Y_2, n_2, \mathbf{u}) = \frac{1}{2}\rho|\mathbf{u}|^2 - W(\rho, \dot{\rho}, Y_2, n_2)$$

Here

$$W(\rho, \dot{\rho}, Y_2, n_2) = \rho\left(Y_2\varepsilon_2(\rho_2) - \frac{3}{2}\alpha_2\frac{\rho_{10}}{\rho}\dot{R}^2\right).$$

Constraints

$$\rho_t + div(\rho\mathbf{u}) = 0,$$
$$\dot{Y}_2 = 0, \quad \dot{n}_2 = 0, \quad Y_2 = \frac{\alpha_2\rho_2}{\rho}, \; n_2 = \frac{N}{\rho}.$$

Governing equations (Iordansky (1960), Kogarko (1961), Van Wijngaarden (1968)).

$$\rho_t + div(\rho \mathbf{u}) = 0,$$

$$(\rho \mathbf{u})_t + div(\rho \mathbf{u} \otimes \mathbf{u} + p\mathbb{I}) = 0,$$

$$p = \rho \frac{\delta \mathbf{W}}{\delta \rho} - \mathbf{W} + \nu \dot{\rho}.$$

Rayleigh-Lamb equation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{p_2(R) - p}{\rho_1} - \frac{4\mu_1}{\rho_{10}}\frac{\dot{R}}{R}.$$

Example

$$p_2(R) = p_0\left(\frac{R_0}{R}\right)^{3\gamma}, \ \gamma > 1.$$

Acoustic properties of the bubbly liquid

$$c_w^2 \approx \frac{\gamma p_0}{\rho_0 \alpha_{20}} \approx 120 \; m/s$$

for air bubbles at $\alpha_{20} = 10^{-2}$.

Discrete bubble size distribution : mass Lagrangian coordinates

$$\tau_t - u_q = 0, \;\; u_t + p_q = 0,$$

$$R_i R_{itt} + \frac{3}{2} R_{it}^2 = \frac{(p_{2i}(R_i) - p)}{\rho_{10}} - \frac{4\mu_i R_{it}}{\rho_{10} R_i}, \;\; i = 1, ..., M,$$

$$Y_{2it} = 0, \;\; n_{it} = 0,$$

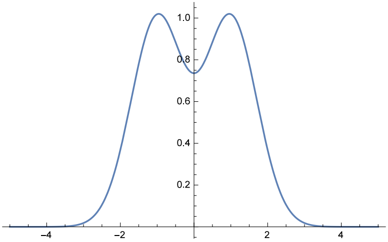
$$\tau = Y_1/\rho_{10} + \frac{4\pi}{3} \sum_{i=1}^{M} n_i R_i^3 \;\; Y_1 + \sum_{i=1}^{M} Y_{2i} = 1.$$

Discrete bubble size distribution : travelling waves

Hamiltonian system with dissipation :

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \ \dot{p}_i = -\frac{\partial H}{\partial q_i} + F_i, \ i = 1, ..., M.$$

Theorem of existence and uniqueness of traveling wave solutions with dissipation effects (SG, Fil'ko, 1991).
No results for Hamiltonian case (M-body problem).
Problem of saddle-center connection, discrete spectrum of wave velocities, multi-hump solitary waves.



Continuous bubble size distribution

$$\tau_t - u_q = 0,\ u_t + p_q = 0,$$

$$RR_{tt} + \frac{3}{2}R_t^2 = \frac{(p_2(\xi, R) - p(t,q))}{\rho_{10}} - \frac{4\mu R_t}{\rho_{10} R},$$

$$\tau = \tau(t,q),\ u = u(t,q),\ p = p(t,q),\ R = R(t,q,\xi),$$

$$p_2(\xi, R) = p_0 \left(\frac{\xi}{R}\right)^{3\gamma}.$$

$$\tau = Y_1/\rho_{10} + \frac{4\pi}{3}\int_{\xi_{min}}^{\xi_{max}} n(\xi) R^3(t,q,\xi) d\xi,\ Y_1 + Y_2 = 0,$$

$$Y_2(q) = \int_{\xi_{min}}^{\xi_{max}} y_2(q,\xi) d\xi.$$

Landau damping

Linearized equations

$$R'_{tt} + \omega^2(\xi) R' = -\frac{p'}{\rho_{10}},$$

$$p'_{qq} + 4\pi \int_{\xi_{min}}^{\xi_{max}} n(\xi) \xi^2 R'_{tt} d\xi = 0.$$

Minnaert frequency

$$\omega_1^2 \leq \omega^2(\xi) = \frac{3\gamma p_0}{\rho_{10}\xi^2} \leq \omega_2^2.$$

Initial conditions :

$$R'(0, q, \xi) = R'_0(q, \xi), \; R'_t(0, q, \xi) = R'_1(q, \xi).$$

Monochromatic signal with $\omega \in (\omega_1, \omega_2)$ is decreasing as $\mathcal{O}(t^{-k})$ (k depends on the smoothness of $n(\xi)$).

Governing equations for "not too dilute" two velocity motion of massless bubbly clouds (SG and Teshukov (2002, 2005), SG and Saurel (2002))

$$L = \frac{1}{2}\alpha_1\rho_1\left(1 - \frac{3}{2}\alpha_2\right)\|\mathbf{u}\|^2 + \frac{\rho_1\alpha_1\alpha_2}{4}\|\mathbf{u} - \mathbf{v}\|^2 + \frac{3}{2}\rho_1\alpha_2\dot{R}^2 - \alpha_2\rho_2\varepsilon_2(R).$$

$$\dot{R} = R_t + \mathbf{v}.\nabla R.$$

More general Lagrangian

$$L = L(\mathbf{j}_1, \mathbf{j}_2, \overline{\rho}_1, \overline{\rho}_2, \mathcal{S}_1, \mathcal{S}_2, N_2, \overline{R^k}, \overline{R^k}_t, \nabla \overline{R^k})$$

Here $\overline{\rho}_a = \alpha_a \rho_a$ are the apparent densities, $\mathcal{S}_a = \overline{\rho}_a \eta_a$ are partial volume entropies, and $\mathbf{j}_a = \overline{\rho}_a \mathbf{u}_a$. N_2 is the number of inclusions characterized by $\overline{R^k}$ (k=1,2,3 correspond to average size, average surface, and average volume of inclusions).

$$\frac{\partial \overline{\rho}_a}{\partial t} + \nabla \cdot (\overline{\rho}_a \mathbf{u}_a) = 0, \quad \frac{\partial \mathcal{S}_a}{\partial t} + \nabla \cdot (\mathcal{S}_a \mathbf{u}_a) = 0, \quad \frac{\partial N_2}{\partial t} + \nabla \cdot (N_2 \mathbf{u}_2) = 0.$$

Micro-inertia equations :

$$\frac{\partial L}{\partial \overline{R^k}} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial (\overline{R^k})_t}\right) - div\left(\frac{\partial L}{\partial \nabla \overline{R^k}}\right) = 0.$$

Numerics : inversion of an elliptic operator (O. Le Metayer, SG & S. Hank (2010))

System to solve :

$$\tau_t - u_q = 0, \; u_t - \left(\frac{\partial \tilde{e}}{\partial \tau} - \frac{\partial}{\partial t} \left(\frac{\partial \tilde{e}}{\partial \tau_t} \right) \right)_q = 0.$$

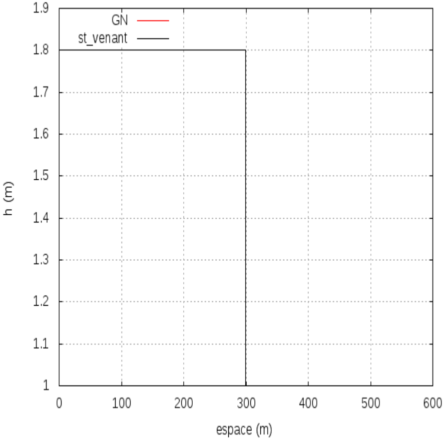
Or

$$\tau_t - u_q = 0, \; K_t - \left(\frac{\partial \tilde{e}}{\partial \tau} \right)_q = 0,$$

$$K = u + \left(\frac{\partial \tilde{e}}{\partial \tau_t} \right)_q = u - \frac{1}{3} \left(\frac{u_q}{\tau^4} \right)_q = \mathcal{A} u.$$

To find u we need to invert operator $\mathcal{A}$.

Dam-break problem : comparison between SGN and Saint-Venant equations



Shortcomings of the method

- Prohibitively expensive computations
- Complicated “transparent” boundary conditions (M. Kazakova, P. Noble, 2018)

Extended Lagrangian approach

From “master” lagrangian to “extended lagrangian” (N. Favrie, SG, 2017, F. Dhaouadi, N. Favrie, SG, 2018).