Locally implicit time schemes for transient visco-elastic wave propagation problems on non-uniform meshes

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Context Elastic and visco-elastic models

Carcione, J.M. (2007)

$$\begin{cases} \dfrac{d^2 u}{dt^2} - \nabla \cdot \boldsymbol{\sigma} = f \\ \text{Inviscid} \quad \boldsymbol{\sigma} = \mathcal{C}\varepsilon(u) \\ \text{Kelvin-Voigt} \quad \boldsymbol{\sigma} = \mathcal{C}^{\mathrm{kv}}\varepsilon(\mathbf{u}) + \mathcal{D}^{\mathrm{kv}}\partial_t\varepsilon(\mathbf{u}) \\ \text{Maxwell} \quad \mathcal{S}\partial_t\boldsymbol{\sigma} + \mathcal{H}\boldsymbol{\sigma} = \partial_t\varepsilon(\mathbf{u}) \\ \text{Zener} \quad \boldsymbol{\sigma} + \tau\partial_t\boldsymbol{\sigma} = \mathcal{C}^{\mathrm{zn}}\varepsilon(\mathbf{u}) + \tau\mathcal{D}^{\mathrm{zn}}\partial_t\varepsilon(\mathbf{u}) \\ \text{+ initial/boundary conditions} \end{cases}$$

Let $(\mathcal{C}_\star , \mathcal{D}_\star , \rho_\star)$ be the mechanical properties at a frequency $\omega_\star$:

- Inviscid $\mathcal{C} = \mathcal{C}_\star$
- Kelvin-Voigt $\mathcal{C}^{\mathrm{kv}} = \mathcal{C}_\star \qquad \mathcal{D}^{\mathrm{kv}} = \frac{1}{\omega_\star}\mathcal{D}_\star$
- Maxwell $\mathcal{S} = (\mathcal{C}^{\mathrm{mx}})^{-1} = (\mathcal{C}_\star + \mathcal{D}_\star\mathcal{C}_\star^{-1}\mathcal{D}_\star)^{-1} \qquad \mathcal{H} = (\mathcal{D}^{\mathrm{mx}})^{-1} = (\frac{1}{\omega_\star}\mathcal{D}_\star + \mathcal{C}_\star\mathcal{D}_\star^{-1}\mathcal{C}_\star)^{-1}$
- Zener $\mathcal{C}^{\mathrm{zn}} = \mathcal{C}_\star - \mathcal{D}_\star \qquad \mathcal{D}^{\mathrm{zn}} = \mathcal{C}_\star + \mathcal{C}\mathcal{D}_\star \qquad \tau = \frac{1}{\omega_\star}$

Imperiale, A., Leymarie N., Demaldent, E. (2020)

Context Conform lumped elements & leapfrog scheme

- Linear high-frequency elastic wave propagation problem: $\forall t \in]0;T]$, find $\mathbf{u}(t) \in \mathbf{V}$ such that $\forall \mathbf{v} \in \mathbf{V}$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} m(\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) = \ell(t;\mathbf{v})$$

$$m(\mathbf{w},\mathbf{v}) = \int_\Omega \varrho \mathbf{w}\cdot\mathbf{v}\,\mathrm{d}\Omega$$

Mass bilinear form

$$a(\mathbf{w},\mathbf{v}) = \int_\Omega (\mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{w})) : \boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}\Omega$$

Stiffness bilinear form

$$\ell(t;\mathbf{v}) = \int_\Gamma \mathbf{g}(t)\cdot\mathbf{v}\,\mathrm{d}\Gamma$$

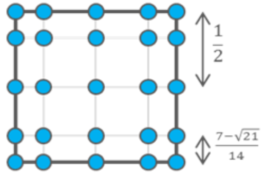
Surface loading (*e.g.* PZT actuator)

- Conform finite element space allowing **consistent mass-lumping** (diagonal mass matrix)

Cohen, G. (2002), Joly, P. (2007), Komatitsch, D. *et al* (1999), Chin-Joe-Kong, M. J. S *et al.* (1999), Mulder, W. A. *et al* (2016), …

$$\mathbf{V}_h = \left\{ v_h \in \mathcal{C}^0(\overline{\Omega}) \mid \forall K \in \mathcal{T}_h,\ \exists! \widehat{v} \in \mathcal{V}(\widehat{K}),\ v_h|_K = \widehat{v} \circ \mathbf{F}_K^{-1} \right\}^N \subset \mathbf{V}$$

e.g. Quad/Hexa spectral elements: $\mathcal{V}(\widehat{K}) = Q^k(\widehat{K})$



Gauss-Lobatto

- Explicit second order scheme (leapfrog scheme): $\forall n \in [\![1;N_T]\!]$, find $\mathbf{u}_h^{n+1} \in \mathbf{V}_h$ such that $\forall \mathbf{v}_h \in \mathbf{V}_h$

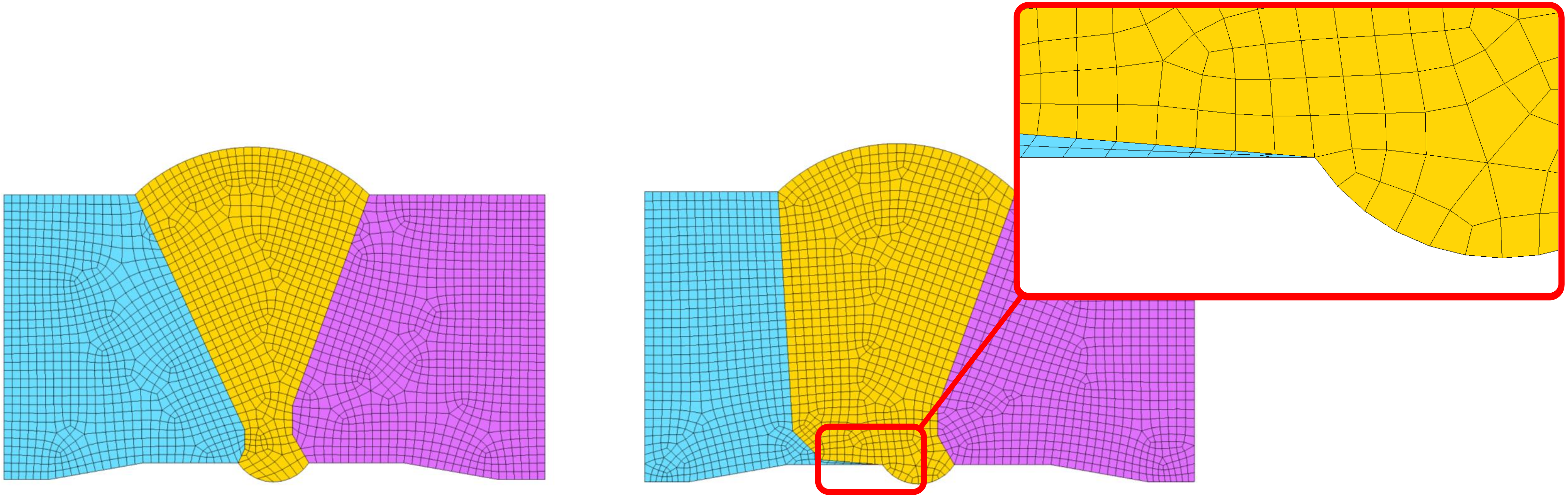
$$\frac{1}{\Delta t^2}\, m(\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \mathbf{v}_h) + a(\mathbf{u}_h^n, \mathbf{v}_h) = \ell(t^n; \mathbf{v}_h)$$

- Stable upon satisfying the CFL condition:

$$\Delta t \leq \overline{\Delta t}_{\mathrm{ex}} = 2 \left(\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{a(\mathbf{v}_h,\mathbf{v}_h)}{m(\mathbf{v}_h,\mathbf{v}_h)} \right)^{-\frac{1}{2}} \quad \Rightarrow$$

Depends on:
- The material (wave(s) speed),
- The type of finite element,
- The quadrature formula,
- **The geometry of the mesh elements.**

Context CFL & geometry of elements



CFL condition : 0,00132348 μs

CFL condition : 0,00037644 μs

Context Im – Ex schemes: enhanced robustness ?

- Can we render a **more robust scheme *w.r.t.* the geometry of the mesh elements ?**

  1. Identifying penalizing terms of the (stiffness) bilinear forms,
  2. Applying a specific time discretization for these terms.

- Two families of specific time discretizations can be identified:

  - Explicit treatment of the penalizing terms *e.g.* **the local time stepping approach**

    Diaz, J. *et al.* (2009), Chabassier, J. *et al* (2019), Carle, C. *et al.* (2020), Grote, M. *et al.* (2021), …

  - Implicit treatment of the penalizing terms leading **to implicit – explicit schemes,** or **locally implicit schemes**

    Rylander, T. *et al.* (2002), Descombes, S. *et al.* (2013), Hochbruck, M. *et al* (2016),…

$\Rightarrow$ Extension of this approach for elastic waves for non-uniform meshes & plate-like geometries, with three-step time schemes

Locally Implicit Scheme

- Decomposition of the bilinear forms in the sum of two non-negative terms: $a(\cdot,\cdot) = (a_\mathrm{c} + a_\mathrm{f})(\cdot,\cdot)$ $\quad m(\cdot,\cdot) = (m_\mathrm{c} + m_\mathrm{f})(\cdot,\cdot)$
- We apply a θ – scheme $\{\mathbf{u}_h^n\}_\theta = \theta \mathbf{u}_h^{n+1} + (1-2\theta)\mathbf{u}_h^n + \theta \mathbf{u}_h^{n-1}$ on the potentially penalizing stiffness term:

$$\frac{1}{\Delta t^2}\, m(\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \mathbf{v}_h) + a_\mathrm{c}(\mathbf{u}_h^n, \mathbf{v}_h) + a_\mathrm{f}(\{\mathbf{u}_h^n\}_\theta, \mathbf{v}_h) = \ell(t^n; \mathbf{v}_h)$$

Lemma. Assuming that $\theta \geq \frac{1}{4}$ the locally implicit scheme is stable upon the following sufficient condition:

$$\Delta t \leq \overline{\Delta t}_{\mathrm{li}} = 2 \left(\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{a_\mathrm{c}(\mathbf{v}_h, \mathbf{v}_h)}{m_\mathrm{c}(\mathbf{v}_h, \mathbf{v}_h)} \right)^{-\frac{1}{2}}$$

Sketch of the proof through energy arguments $\left(\ell(t^n, \cdot) = 0\right)$

- Remark that $\mathbf{u}_h^n = \{\mathbf{u}_h^n\}_{1/4} - \frac{\Delta t^2}{4}\left(\frac{\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}}{\Delta t^2}\right)$
- Using as test function $\mathbf{v}_h = \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\Delta t}$ leads to the conservation of the energy functional

$$\mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2}\widetilde{m}\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}\right) + \frac{1}{2} a\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right)$$

- The modified kinetic term reads

$$\widetilde{m}(\cdot,\cdot) = \left(m - \frac{\Delta t^2}{4} a_\mathrm{c} + \Delta t^2(\theta - \tfrac{1}{4}) a_\mathrm{f}\right)(\cdot,\cdot)$$

- Since $\theta \geq \frac{1}{4}$ and $a_\mathrm{f}(\cdot,\cdot) \geq 0$, the conserved functional is positive if

$$\Delta t \leq 2 \left(\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{a_\mathrm{c}(\mathbf{v}_h, \mathbf{v}_h)}{m(\mathbf{v}_h, \mathbf{v}_h)} \right)^{-\frac{1}{2}}$$

- We conclude using $m(\cdot,\cdot) \geq m_\mathrm{c}(\cdot,\cdot)$.

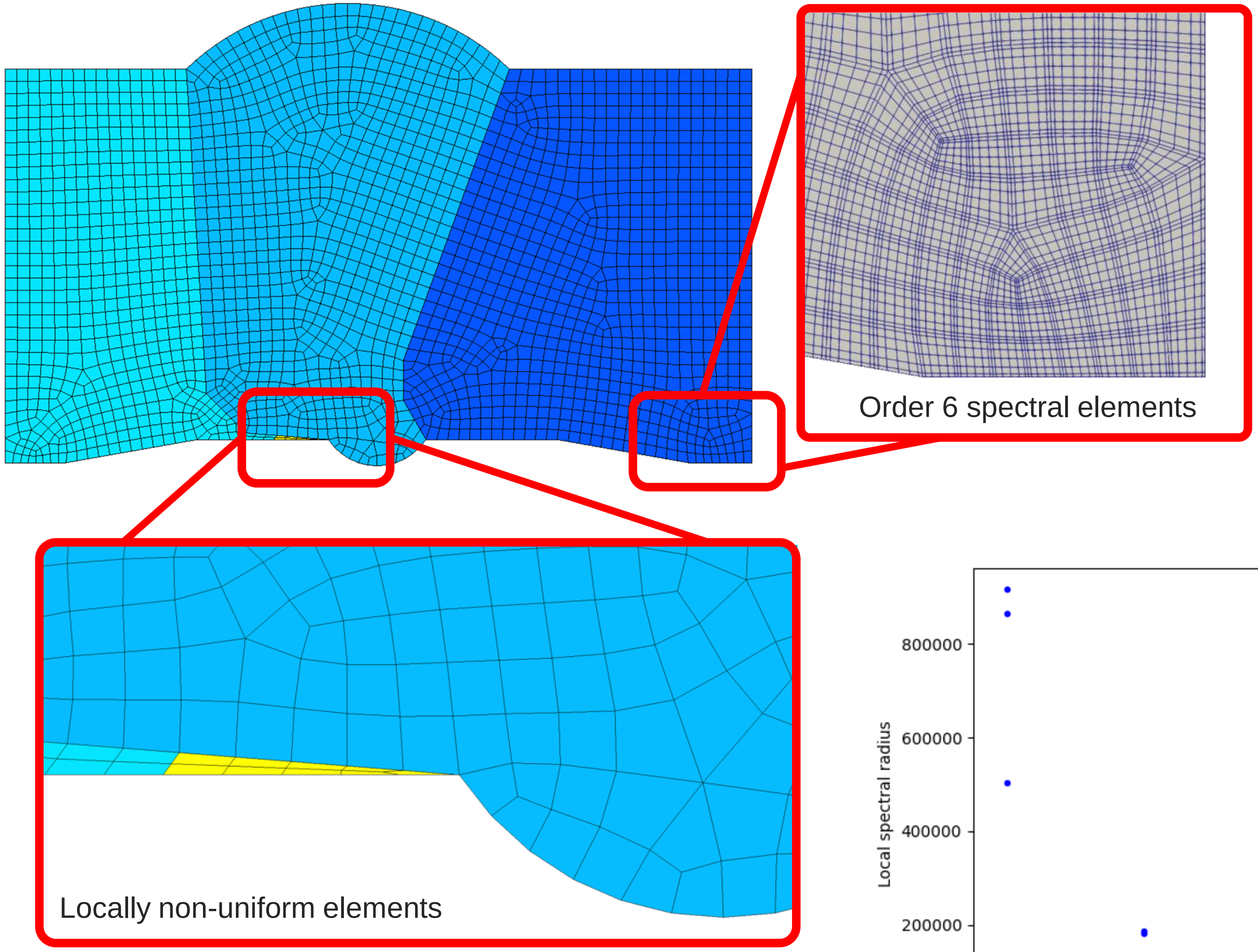
Locally Implicit Scheme

- Decomposition of the bilinear forms in the sum of two non-negative terms: $a(\cdot,\cdot) = (a_c + a_f)(\cdot,\cdot) \qquad m(\cdot,\cdot) = (m_c + m_f)(\cdot,\cdot)$
- We apply a θ – scheme $\{\mathbf{u}_h^n\}_\theta = \theta\mathbf{u}_h^{n+1} + (1-2\theta)\mathbf{u}_h^n + \theta\mathbf{u}_h^{n-1}$ on the potentially penalizing stiffness term:

$$\frac{1}{\Delta t^2} m(\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \mathbf{v}_h) + a_c(\mathbf{u}_h^n, \mathbf{v}_h) + a_f(\{\mathbf{u}_h^n\}_\theta, \mathbf{v}_h) = \ell(t^n; \mathbf{v}_h)$$

$$(\frac{1}{\Delta t^2}\mathbb{M} + \theta\mathbb{K}_f)\mathbf{u}^{n+1} = l(t^n) - (\mathbb{K}_c + (1-2\theta)\mathbb{K}_f - \frac{2}{\Delta t^2}\mathbb{M})\mathbf{u}^n - (\frac{1}{\Delta t^2}\mathbb{M} + \theta\mathbb{K}_f)\mathbf{u}^{n-1}$$

Selection process

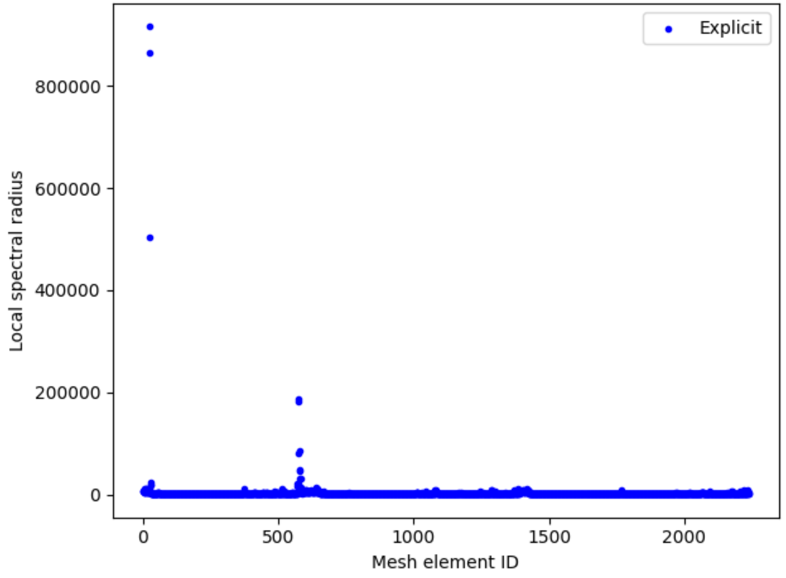


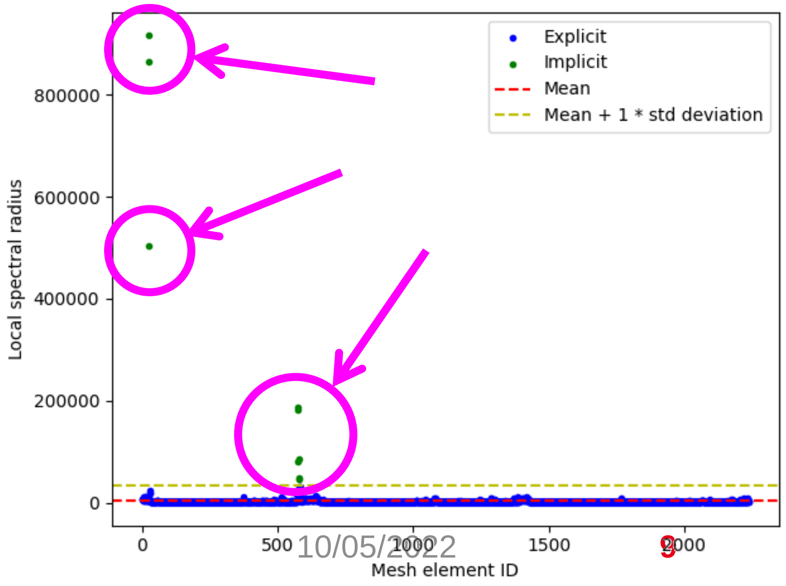
- Using the "*Irons & Treharne*" theorem

Cottereau, R. *et al.* (2018) (& references therein)

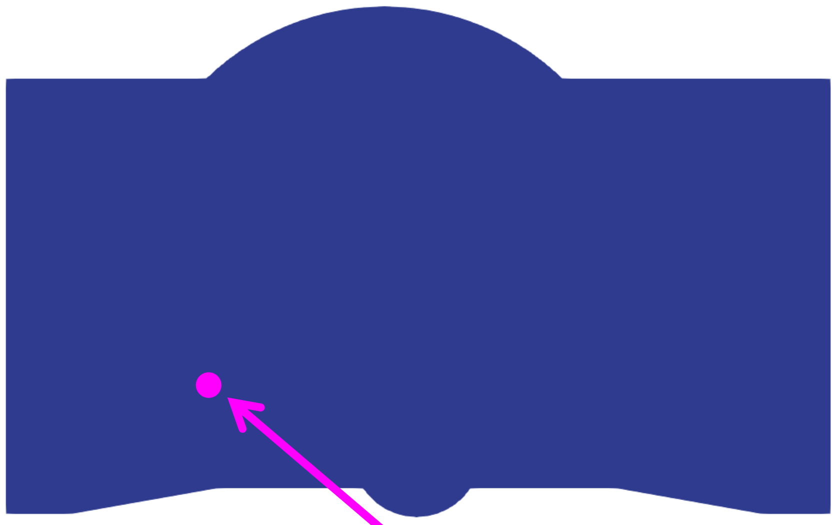
$$2\left(\max_{K\in\mathcal{T}_h}\ \sup_{\mathbf{v}_h\in\mathcal{V}(K)^N}\frac{a_c(\mathbf{v}_h,\mathbf{v}_h)}{m_c(\mathbf{v}_h,\mathbf{v}_h)}\right)^{-\frac{1}{2}}\leq\overline{\Delta t}_{\mathrm{li}}$$

We analyse **"local" Rayleigh quotients** to identify penalizing elements



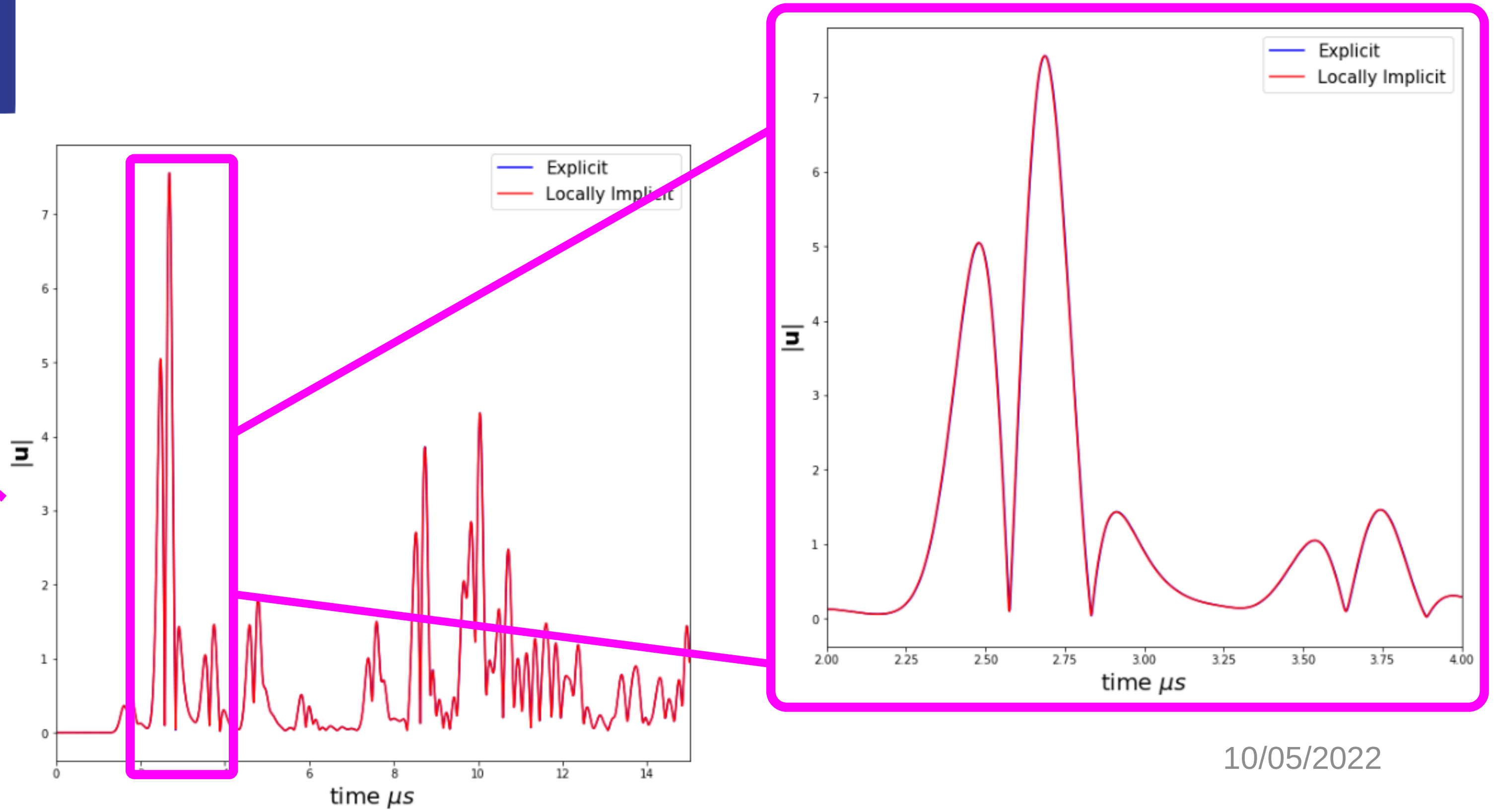


2D Test Cases "Unfortunate" CAO

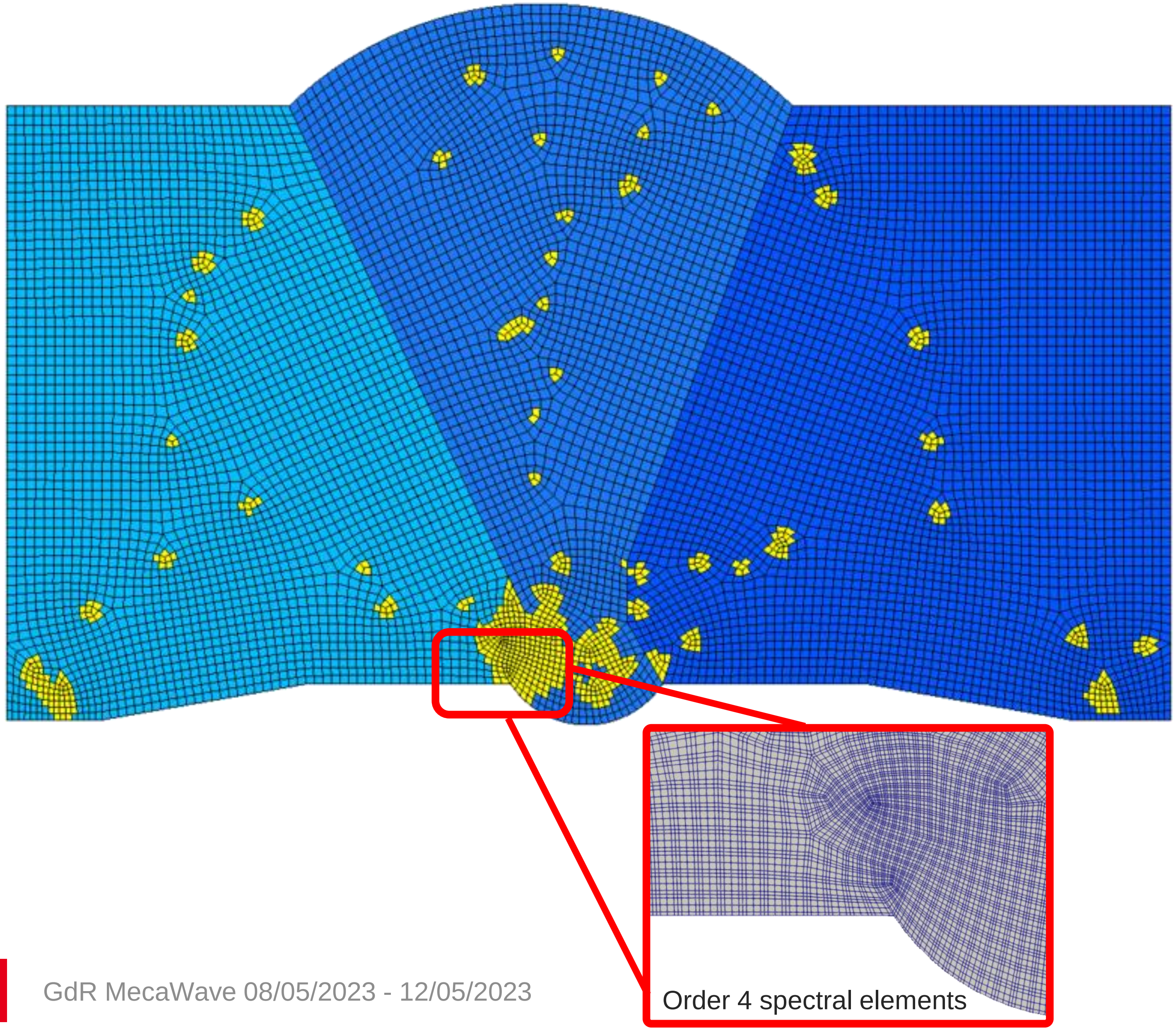


	$\Delta t(\mu s)$	CPU^* (s)	$\frac{\Delta t_{IMEX}}{\Delta t_{EX}}$	$\frac{CPU^*_{EX}}{CPU^*_{IMEX}}$
EX.	0.000338801	450.6		
IM.EX. (1)	0.0020605	77.9	6.1	5.8
IM.EX. (2)	0.00165706	85.2	4.8	5.3
IM.EX. (3)	0.00125674	139.2	3.7	3.2
IM.EX. (4)	0.00125674	111.6	3.7	4.0

* on a laptop PC, Intel(R) Core(TM) i7-8850H CPU @ 2.60 GHz

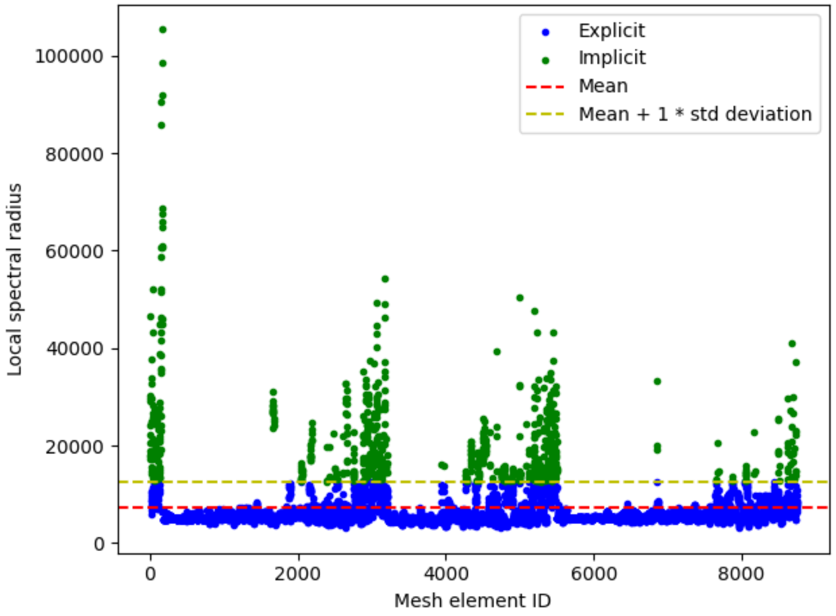


2D Test Cases Quadrilateral mesh

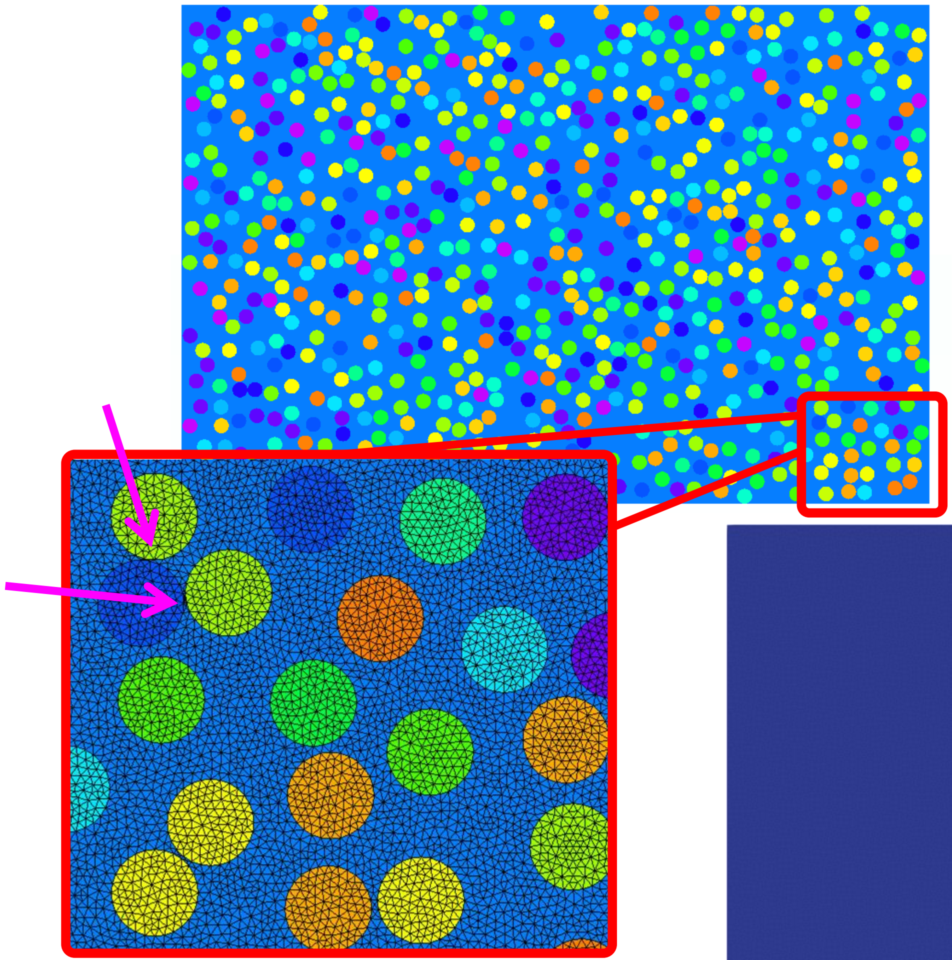


	$\Delta t (\mu s)$	CPU* (s)	$\frac{\Delta t_{IMEX}}{\Delta t_{EX}}$	$\frac{CPU^*_{EX}}{CPU^*_{IMEX}}$
EX.	0.00232952	93.7		
IM.EX. (1)	0.00654526	62.6	2.8	1.5
IM.EX. (2)	0.00560053	58.7	2.4	1.6
IM.EX. (3)	0.00494477	54.5	2.1	1.7
IM.EX. (4)	0.00457334	45.2	2.0	2.0

* on a laptop PC, Intel(R) Core(TM) i7-8850H CPU @ 2.60 GHz



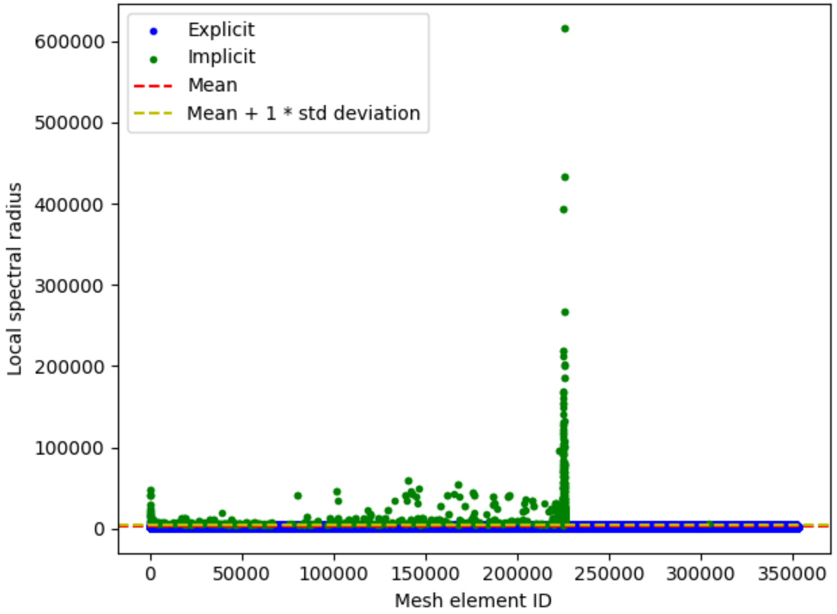
2D Test Cases Arbitrary close multiple (~700) inclusions in isotropic material



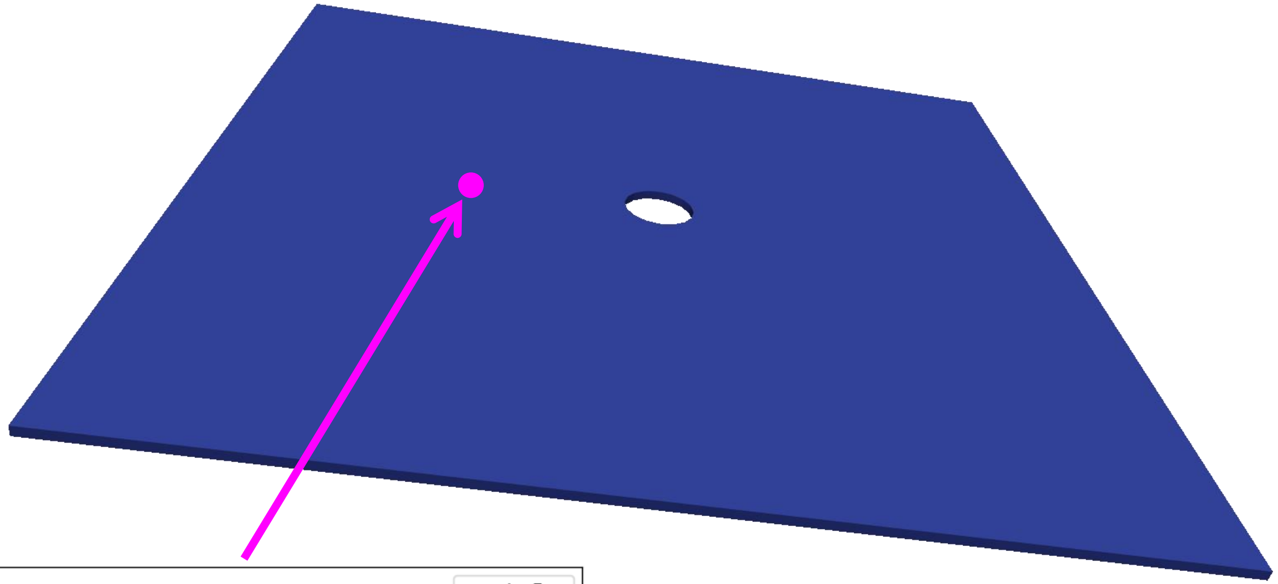
P2 & bubble triangular finite elements

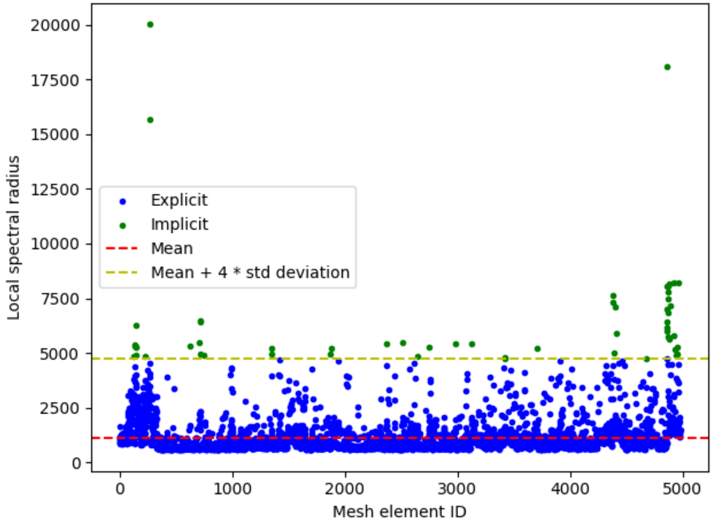
	$\Delta t(\mu s)$	CPU^* (s)	$\frac{\Delta t_{IMEX}}{\Delta t_{EX}}$	$\frac{CPU^*_{EX}}{CPU^*_{IMEX}}$
EX.	0.00466549	1040.5		
IM.EX. (1)	0.039968	136.3	8.6	7.6
IM.EX. (2)	0.033219	151.8	7.12	6.8
IM.EX. (3)	0.0297385	151.2	6.4	6.8
IM.EX. (4)	0.0264332	177.6	5.7	5.8

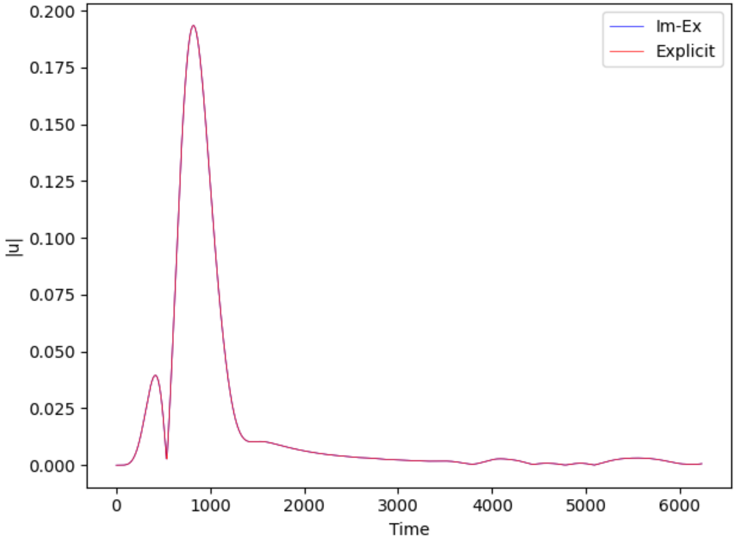
* on a laptop PC, Intel(R) Core(TM) i7-8850H CPU @ 2.60 GHz



3D Test Cases Plate with Hole

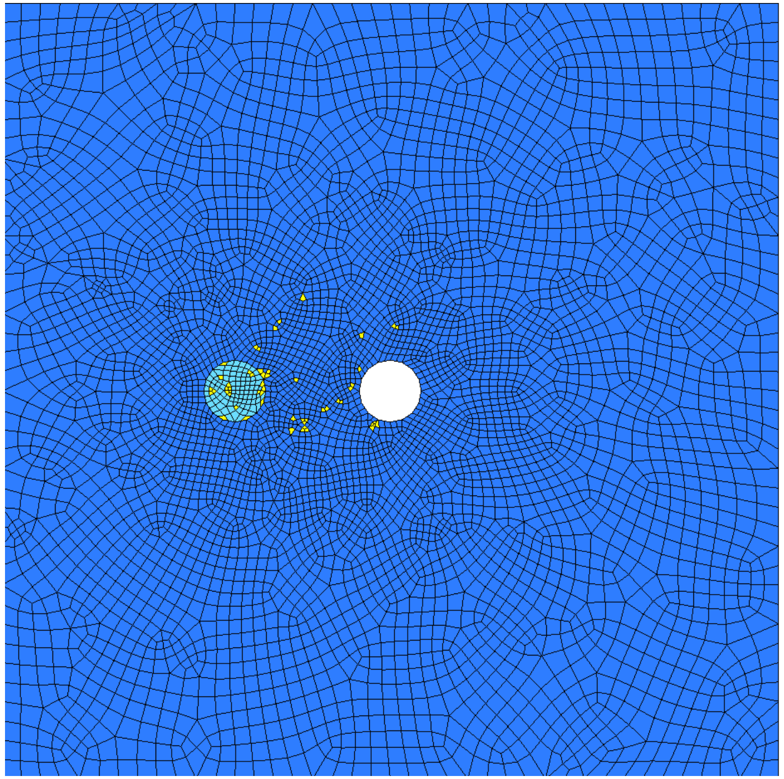




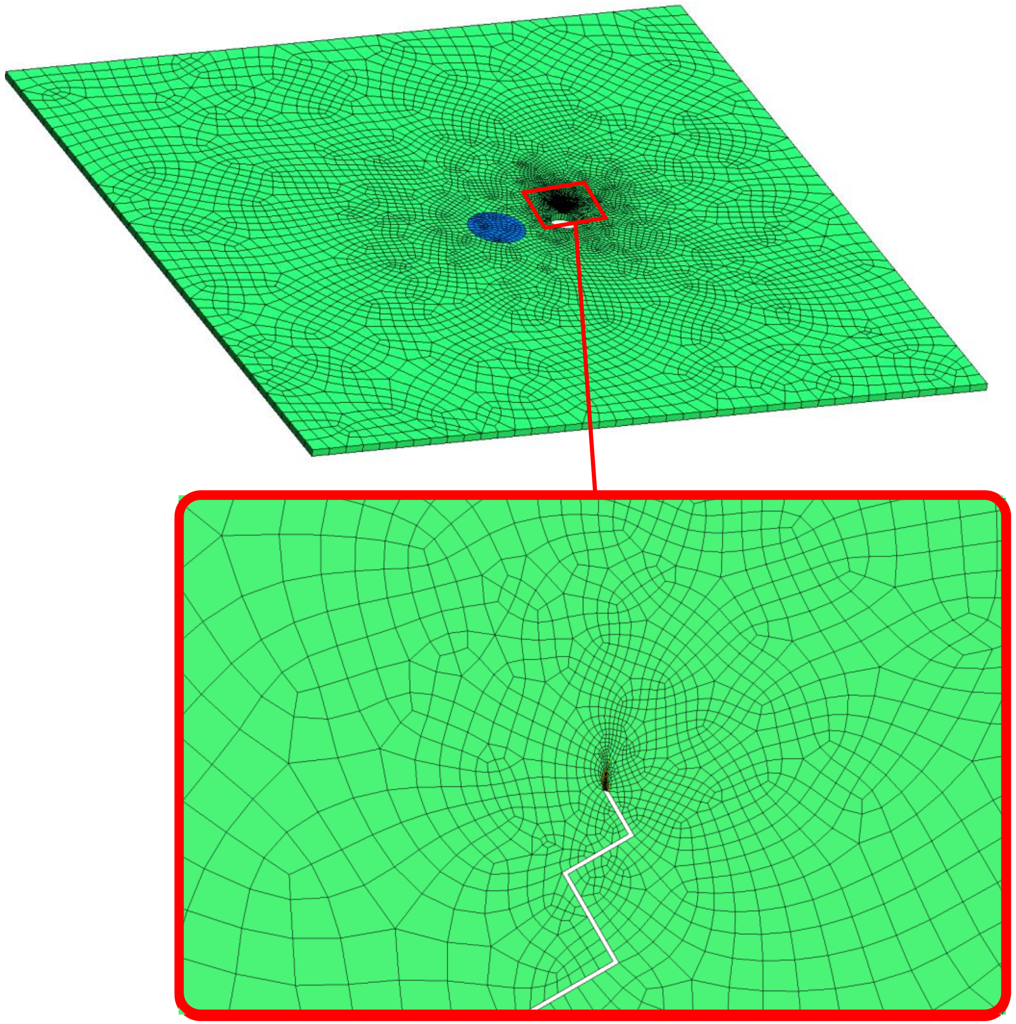


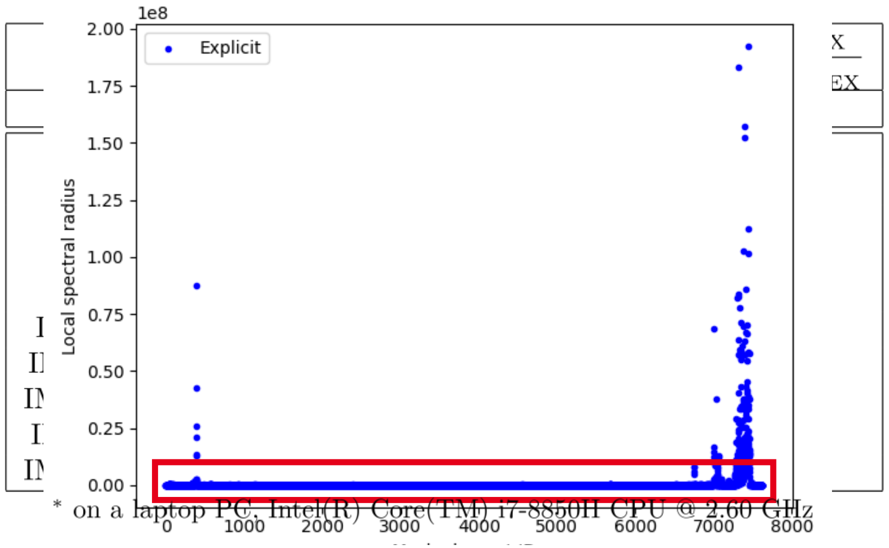
	$\Delta t (\mu s)$	CPU* (s)	$\frac{\Delta t_{\text{IMEX}}}{\Delta t_{\text{EX}}}$	$\frac{\text{CPU}^*_{\text{EX}}}{\text{CPU}^*_{\text{IMEX}}}$
EX.	0.0160424	64,4		
IM.EX. (1)	0.0481272	81.2	3	0.8
IM.EX. (2)	0.0320848	79.0	2	0.8
IM.EX. (3)	0.0320848	52.6	2	1.2
IM.EX. (4)	0.0320848	42.6	2	1.5

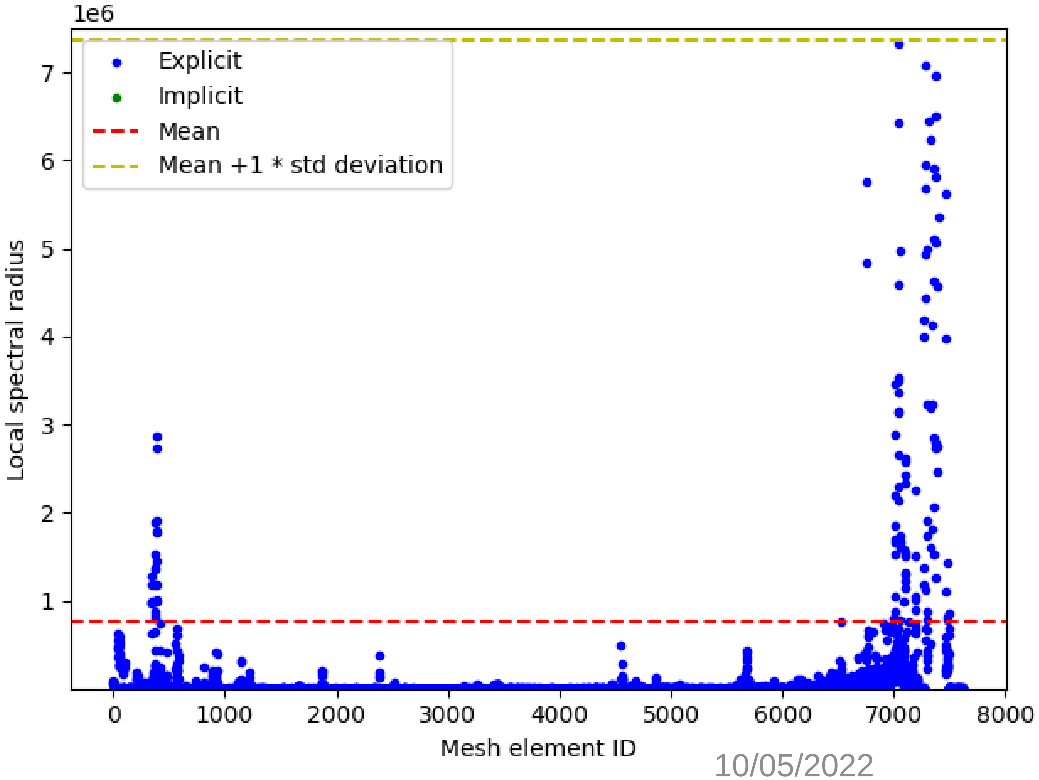
* on a laptop PC, Intel(R) Core(TM) i7-8850H CPU @ 2.60 GHz



3D Test Cases Plate with Hole and Crack







Extension to Visco-Elastic models

$\forall n \in [\![0; N_t]\!]$, find $(\mathbf{u}_h^{n+1}, \boldsymbol{\sigma}_h^{n+1}) \in \mathbf{V}_h \times \mathbf{W}_h$ such that $\forall(\mathbf{v}_h, \boldsymbol{\mu}_h) \in \mathbf{V}_h \times \mathbf{W}_h$ we have

Maxwell Model (primal-dual)

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}t^2} m_\rho^V(\mathbf{u}, \mathbf{u}^*) + b(\boldsymbol{\sigma}, \mathbf{u}^*) = 0\,, \\ \frac{\mathrm{d}}{\mathrm{d}t} m^W[\mathcal{S}](\boldsymbol{\sigma}, \boldsymbol{\sigma}^*) + m^W[\mathcal{H}](\boldsymbol{\sigma}, \boldsymbol{\sigma}^*) - \frac{\mathrm{d}}{\mathrm{d}t} b(\boldsymbol{\sigma}^*, \mathbf{u}) = 0\,, \end{cases}$$

$\mathbf{V} = [H^1(\Omega)]^d$

$\mathbf{W} = [L^2(\Omega)]^{d'}$

$m_\alpha^V(\mathbf{u}, \mathbf{u}^*) = \int_\Omega \alpha \mathbf{u} \cdot \mathbf{u}^* \mathrm{d}\Omega\,,\ \forall \mathbf{u}, \mathbf{u}^* \in \mathbf{V}$

$m^W[\mathcal{A}](\sigma, \sigma^*) = \int_\Omega \mathcal{A}\sigma : \sigma^* \mathrm{d}\Omega\,,\ \forall \sigma, \sigma^* \in \mathbf{W}$

$b(\sigma^*, \mathbf{u}^*) = \int_\Omega \sigma^* : \boldsymbol{\varepsilon}(\mathbf{u}^*) \mathrm{d}\Omega\,,\ \forall (\mathbf{u}^*, \sigma^*) \in \mathbf{V} \times \mathbf{W}$

- Decomposition of the stress constraint into two fields: $\boldsymbol{\sigma}_h^n = \boldsymbol{\sigma}_{h,c}^n + \boldsymbol{\sigma}_{h,f}^n$.
- Decomposition of the bilinear forms into two terms: $b(\cdot\,,\cdot) = b_c(\cdot\,,\cdot) + b_f(\cdot\,,\cdot)$.
- Decomposition of the bilinear form $m(\cdot\,,\cdot)$ into two symmetric positive terms: $m(\cdot\,,\cdot) = m_c(\cdot\,,\cdot) + m_f(\cdot\,,\cdot)$.
- We apply a θ – scheme $\{\boldsymbol{\sigma}_h^n\}_\theta = \theta\boldsymbol{\sigma}_h^{n+1} + (1-2\theta)\boldsymbol{\sigma}_h^n + \theta\boldsymbol{\sigma}_h^{n-1}$ on the potentially penalizing stiffness term:

Locally Implicit Scheme

$$\begin{cases} \frac{1}{\Delta t^2} m_\rho^V(\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \mathbf{v}_h) + b_c(\sigma_{h,c}^n, \mathbf{v}_h) + b_f(\{\sigma_{h,f}^n\}_\theta, \mathbf{v}_h) = 0\,, \\ m^W[\mathcal{S}](\frac{\boldsymbol{\sigma}_{h,c}^{n+1} - \boldsymbol{\sigma}_{h,c}^n}{\Delta t}, \boldsymbol{\mu}_{h,c}) + m^W[\mathcal{H}](\frac{\boldsymbol{\sigma}_{h,c}^{n+1} + \boldsymbol{\sigma}_{h,c}^n}{2}, \boldsymbol{\mu}_{h,c}) - b_c(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \boldsymbol{\mu}_{h,c}) = 0\,, \\ m^W[\mathcal{S}](\frac{\boldsymbol{\sigma}_{h,f}^{n+1} - \boldsymbol{\sigma}_{h,f}^{n-1}}{2\Delta t}, \boldsymbol{\mu}_{h,f}) + m^W[\mathcal{H}](\{\boldsymbol{\sigma}_{h,f}^n\}_\theta, \boldsymbol{\mu}_{h,f}) - b_f(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\Delta t}, \boldsymbol{\mu}_{h,f}) = 0\,. \end{cases}$$

Extension to Visco-Elastic models

Stability Condition through Energy Arguments

Lemma. Assuming that $\theta \geq \frac{1}{4}$ the locally implicit scheme is stable upon the following sufficient condition:

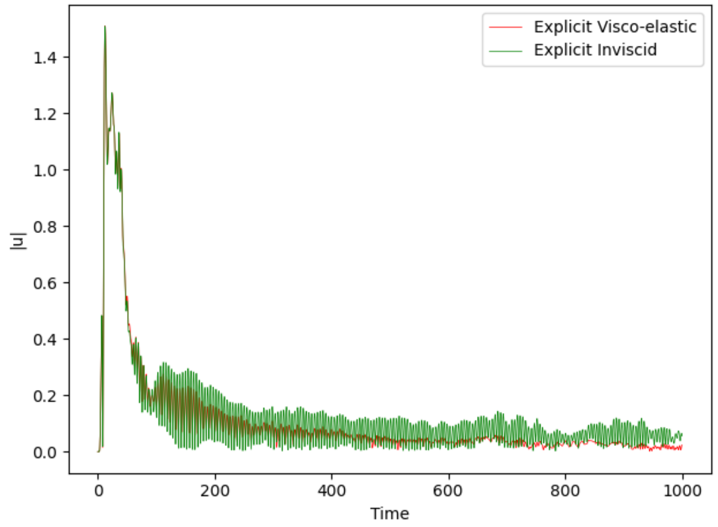
$$\Delta t \leq 2 \left(\sup_{\mathbf{v}_h \in \mathbf{v}_h} \frac{a_c[\mathcal{C}^{\mathrm{mx}}](\mathbf{v}_h, \mathbf{v}_h)}{m^V_{\rho,c}(\mathbf{v}_h, \mathbf{v}_h)} \right)^{-\frac{1}{2}} . \qquad \mathcal{C}^{\mathrm{mx}} = \mathcal{C}_\star + \mathcal{D}_\star \mathcal{C}_\star^{-1} \mathcal{D}_\star$$

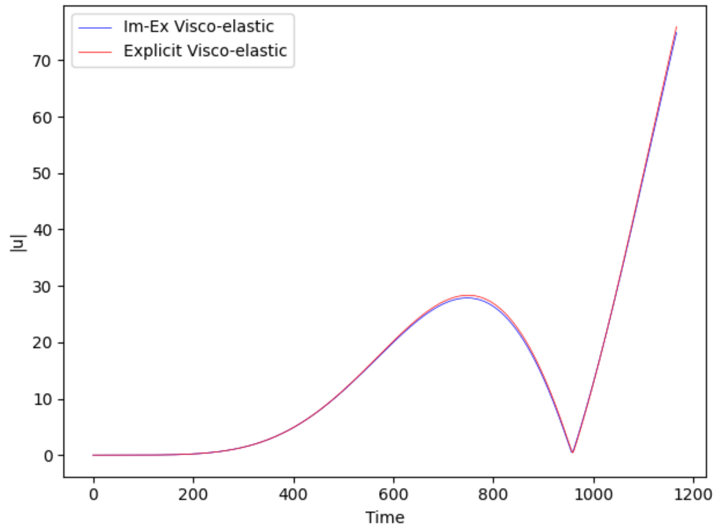
- Take $\mathbf{v}_h = \frac{1}{2\Delta t}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1})$ as test function in first equation and remark that $\boldsymbol{\sigma}_h^n = \{\boldsymbol{\sigma}_h^n\}_{1/4} + (\theta - \frac{1}{4})(\boldsymbol{\sigma}_h^{n+1} - 2\boldsymbol{\sigma}_h^n + \boldsymbol{\sigma}_h^{n-1})$.
- Take $\boldsymbol{\mu}_{h,c} = \frac{1}{2}(\boldsymbol{\sigma}_h^{n+1} + \boldsymbol{\sigma}_h^n)$ as test function in second equation, then compute the average at times $t^{n+\frac{1}{2}}$ and $t^{n-\frac{1}{2}}$.
- Take $\boldsymbol{\mu}_{h,f} = \{\boldsymbol{\sigma}_h^n\}_\theta$ as test function in third equation.
- Summing obtained equations leads to the following energy functional :

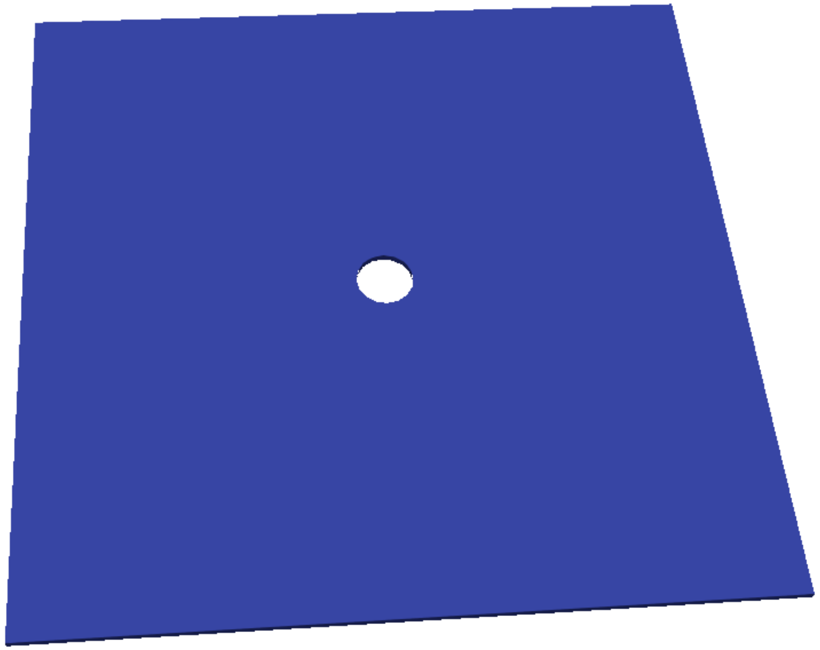
$$\begin{aligned}\mathcal{E}_h^{n+\frac{1}{2}} =& m^W[\mathcal{S}]\left(\frac{\boldsymbol{\sigma}_h^{n+1} + \boldsymbol{\sigma}_h^n}{2}, \frac{\boldsymbol{\sigma}_h^{n+1} + \boldsymbol{\sigma}_h^n}{2}\right) + \frac{1}{2}(\theta - 1/4) m^W[\mathcal{S}]\left(\frac{\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n}{2}, \frac{\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n}{2}\right) \\ &+ \frac{1}{2}\left(m_\rho^V - \frac{\Delta t^2}{4} a_c[\mathcal{C}^{\mathrm{mx}}]\right)\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}\right) \\ &+ \frac{\Delta t^2}{8}\left(m^W[\mathcal{S}]\left(\frac{\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n}{2}, \frac{\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n}{2}\right)^{\frac{1}{2}} - a_c[\mathcal{C}^{\mathrm{mx}}]\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}\right)^{\frac{1}{2}} \right)^2\end{aligned}$$

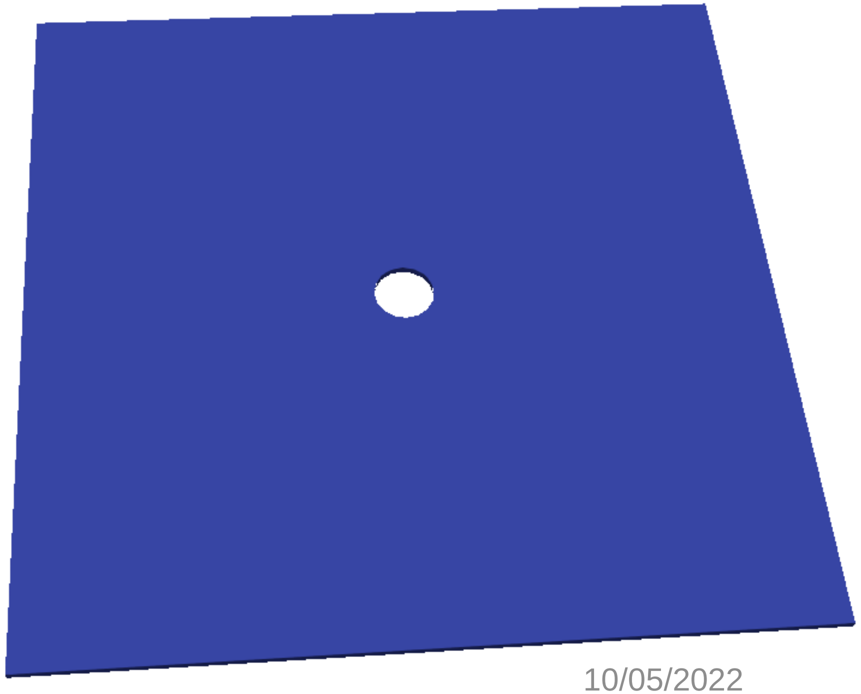
- Since $\theta \geq \frac{1}{4}$ and $m^W[\mathcal{S}](\cdot\,,\cdot)$ is positive, the energy functional is positive if the above condition is satisfied.
- Notice that for any test function $\mathbf{v}_h$, $m_\rho^V(\mathbf{v}_h, \mathbf{v}_h) \geq m^V_{\rho,c}(\mathbf{v}_h, \mathbf{v}_h)$ and conclude.

3D Test Cases









Perspectives

- Complete automatization of selection process
  - Automatically determine optimal factor aka *”sweet spot”*
  - Through assimilation to geometric descriptors
- Convergence analysis
- Combine with Mortar Domain Decomposition
- Local time-stepping
- Optimization through profiling
  - Approximate inverse matrix ?

Extras - Selection process

Types of tagger

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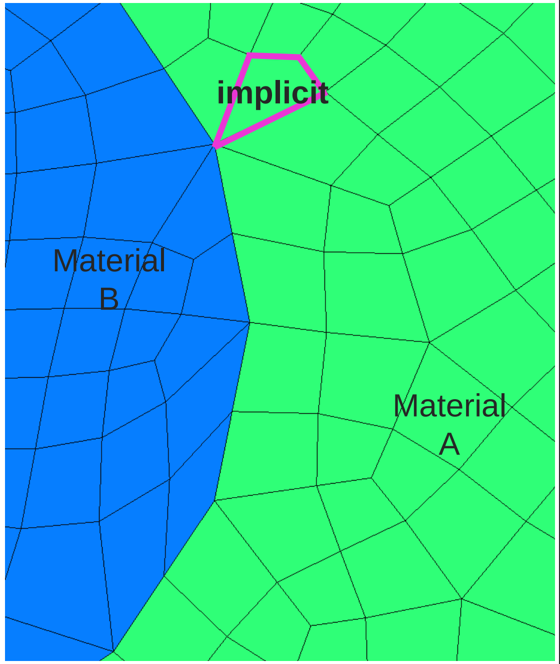
<td>
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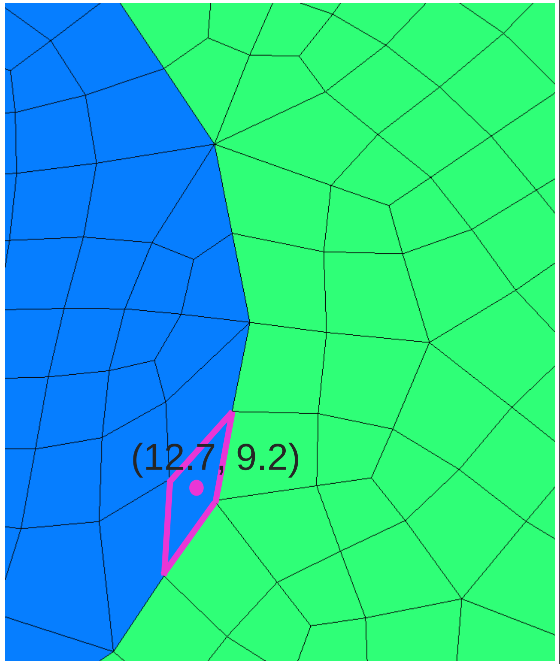
<td>
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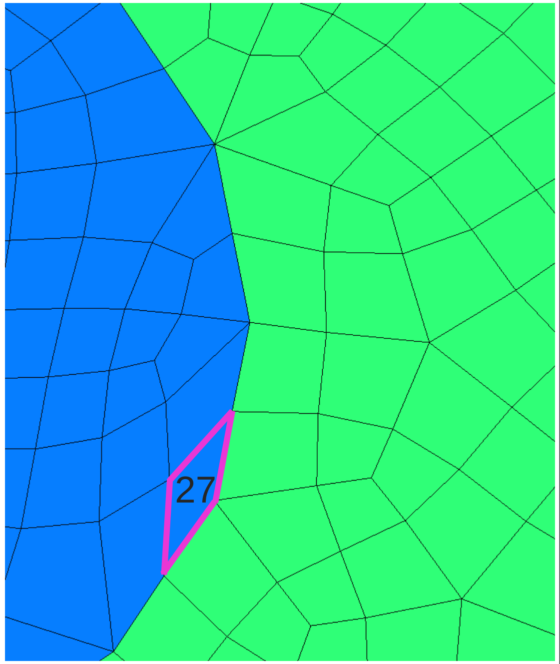
<td>
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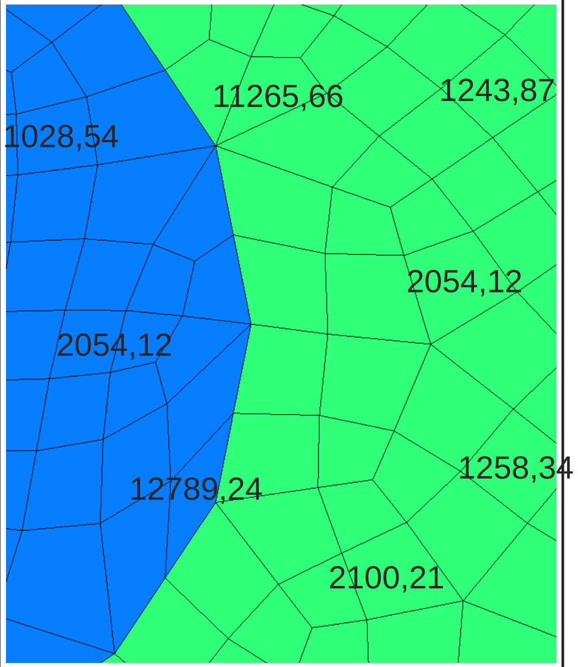
<td>
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