Two-dimensional elastic Bloch waves in helical periodic structures

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Context

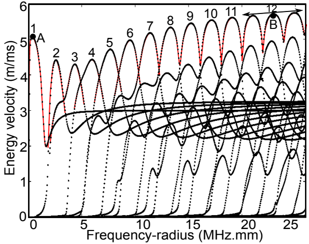
Guided wave applications:

- dynamic analysis of elongated structures
- propagation over long distances, sensitivity to small damages
- examples:
  - **N**on **D**estructive **E**valuation (ultrasonics)
  - vibration and noise reduction
  - statistical energy analysis...

NDE: detecting a damage with elastic waves

Generality about waveguides:

- Guided wave propagation: dispersive and multimodal
- Dispersion curves required
- Modeling tools needed



Energy velocity vs. frequency in a cylindrical bar

On the modeling of elastic waveguides

Full 3D approach:

- high frequency (*e.g.* ultrasonics) → fine mesh
- guided waves go to infinity → large model
- huge computational memory required
- tedious post-processing for wave modes...

Reduced modeling (modal approach):

- guided waves = modes
- eigenvalue problem
- plates, cylinders: analytical approaches (Thomson-Haskell, ©Disperse, ...)
- **arbitrary geometry:** finite element discretization
  - of cross-section[1] (often referred to as "SAFE" method)
  - of a 3D slice with Bloch-Floquet periodic conditions[2] ("**WFEM**")

SAFE: Semi-Analytical Finite Element method
WFEM: **W**ave **F**inite **E**lement **M**ethod

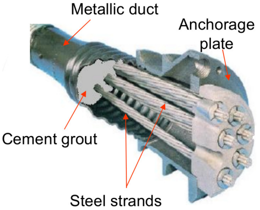
[1] Lagasse JASA 1973, Aalami JAM 1973, Hayashi et al. Ultrasonics 2003, Bartoli et al. JSV 2006,...
[2] Gry et al. JSV 1997, Mace et al. JASA 2005,...

Our motivation: helical structures

Typical example of application: NDT and SHM of cables (damage detection, tension estimation,...)

Helical symmetry:

- along one direction (beam-like structures) or two directions (tube-like)
- continuous (uniform waveguides) or discrete (periodic waveguides)

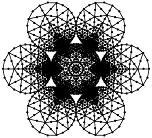


Left: bridge cable (anchorage), **right:** seven-wire strand

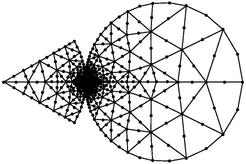
Umbilical power cable

State-of-the-art (GeoEND lab)

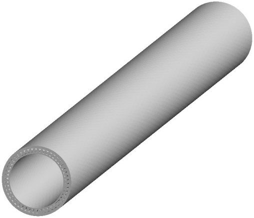
From a 3D seven-wire strand...



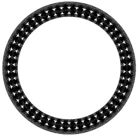
... to SAFE **2D**
(continuous helical symmetry,
12369 dofs)



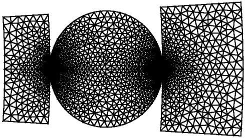
... and to SAFE **2D/6**
(+discrete rot. symmetry,
2094 dof $\rightarrow$ CPU time/13)



From a 3D cable armor...



... to SAFE 2D
(1,000,000 dofs)



... and to **2D/50**
(22587 dofs)

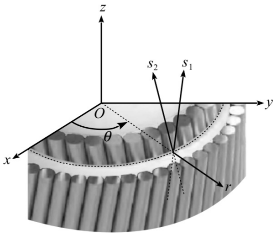
Let us further generalize...

Doubled armored cable (high strength submarine power cable) source: BPP catalog

Double armor

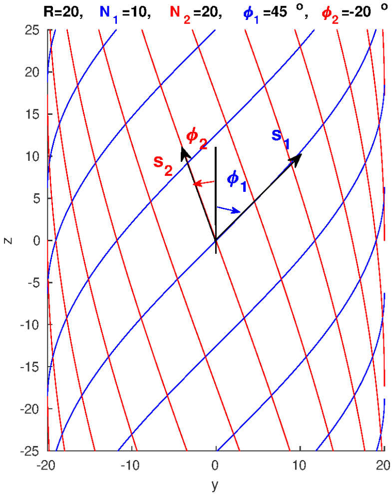
Discrete symmetry along two directions ? (both helical)

Helical coordinates



Cartesian, cylindrical and helical coordinates

- Two layers (virtual or not), Layer 1 (blue) and Layer 2 (red)
- Layer α is divided by N_α helices, oriented along s_α, of radius R_α and lay angle ϕ_α (α=1,2)



Bi-helical periodicity pattern (s_1,s_2): **helical coordinates**

Bi-helical coordinate system

- Relationship between cylindrical and helical coordinates:

$$\begin{cases} \theta = \frac{2\pi}{\ell_1} s_1 + \frac{2\pi}{\ell_2} s_2 \\ z = \frac{L_1}{\ell_1} s_1 + \frac{L_2}{\ell_2} s_2 \end{cases}$$

$L_{1,2}$: helix steps measured along z-axis (straight)

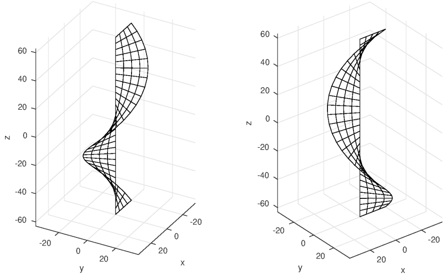
$\ell_{1,2}$: curvilinear steps

- Position vector $\mathbf{OM} = r\mathbf{e}_r(\theta) + z\mathbf{e}_z$:

$$\mathbf{OM} = r\cos\left(\frac{2\pi}{\ell_1} s_1 + \frac{2\pi}{\ell_2} s_2\right)\mathbf{e}_x + r\sin\left(\frac{2\pi}{\ell_1} s_1 + \frac{2\pi}{\ell_2} s_2\right)\mathbf{e}_y + \left(\frac{L_1}{\ell_1} s_1 + \frac{L_2}{\ell_2} s_2\right)\mathbf{e}_z$$

The 3D coordinates are now:

$$(s_1, s_2, r)$$



Surfaces s_2=cst (left), s_1=cst (right)

Unit cell boundaries

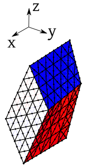
A first difficulty...

Left/right boundaries $\Gamma_1^{\pm}$:
$s_1 = \text{cst} = \pm\Delta\ell_1/2$ (helicoids)

Bottom/top boundaries $\Gamma_2^{\pm}$:
$s_2 = \text{cst} = \pm\Delta\ell_2/2$ (helicoids)

with cell lengths defined by:
$\Delta\ell_1 = \frac{\ell_1}{N_2}\frac{L_2}{L_2 - L_1}, \Delta\ell_2 = \frac{\ell_2}{N_1}\frac{L_1}{L_2 - L_1}$

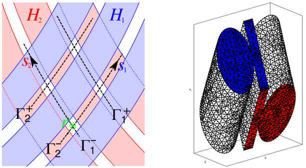
$\rightarrow$ **Unit cells are cut by non-plane surfaces**



Unit cell for a thick hollow cylinder

Parametrization of boundaries for a double armor:

- intersection between 4 helicoids $\Gamma_\alpha^{\pm}$ (2D) and the helical wire volumes H_α (3D)...
- the solution is not analytic
  $\rightarrow$ multiple non-linear equations to solve (in one variable)



Unit cell for a double armor
left: front view, right: FE mesh

Invariance(*) in 2D periodic media

Reminder: Bloch waves in two directions s_1 and s_2: $\psi(s_1, s_2) = e^{ik_1 s_1} e^{ik_2 s_2} u(s_1, s_2)$, with k_α: wavenumbers, u: $\Delta\ell_\alpha$-periodic function ($\Delta\ell_\alpha$: unit cell lengths)

Question: existence of Bloch waves in such a geometry ?

- The coeff. of diff. equations must be periodic in two curved directions
- Let us rewrite the equilibrium equations in the bi-helical system...

Example: equilibrium equation for elasticity

$$\sigma^{ij}_{,j} + \Gamma^i_{mj}\sigma^{mj} + \Gamma^j_{mj}\sigma^{im} + \rho\omega^2 g^{ij} u_j = 0, \text{ with } \sigma^{ij} = C^{ijkl}\epsilon_{kl},\ \epsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k} - \Gamma^m_{kl} u_m)$$

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The coefficients are functions of: the physical properties (ρ and C^{ijkl})... and Γ^k_{ij}!

Key formula: $\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) \rightarrow$ functions of the g_{ij}'s only

Conditions for invariance in periodic media along curved directions (s_1,s_2):

1. the shape of the geometry is ($\Delta\ell_1$,$\Delta\ell_2$)-periodic along (s_1,s_2)
2. the physical properties are ($\Delta\ell_1$,$\Delta\ell_2$)-periodic along (s_1,s_2)
3. **the metric tensor g does not depend on (s_1,s_2)**

(*) "invariance" $\equiv$ the coefficients of the differential equations are periodic

Proof of existence in bi-helical structures

Application to the bi-helical coordinates: let us calculate the metric tensor...

Reminder: $(\mathbf{g})_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$, with $\begin{cases} \mathbf{g}_1 = \frac{\partial \mathbf{OM}}{\partial s_1} \\ \mathbf{g}_2 = \frac{\partial \mathbf{OM}}{\partial s_2} \\ \mathbf{g}_3 = \frac{\partial \mathbf{OM}}{\partial r} \end{cases}$

$$\Rightarrow \mathbf{g} = \begin{bmatrix} \frac{4\pi^2 r^2 + L_1^2}{\ell_1^2} & \frac{4\pi^2 r^2 + L_1 L_2}{\ell_1 \ell_2} & 0 \\ \frac{4\pi^2 r^2 + L_1 L_2}{\ell_1 \ell_2} & \frac{4\pi^2 r^2 + L_2^2}{\ell_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ **point 3 checked** ✓

Corollary: proof of the existence of Bloch waves in bi-helical waveguides whatever the physics

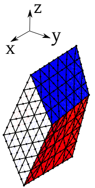
Implementation: the Wave Finite Element Method

FE discretization of the unit cell:

$$(\mathbf{K} - \omega^2\mathbf{M} - \mathrm{i}\omega\mathbf{C})\mathbf{U} = \mathbf{F}$$

Elastic fields U and F are not scalar: Cartesian components must be transformed in the covariant basis

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = \mathbb{J}^{\mathsf{T}} \begin{bmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{bmatrix}, \quad \mathbb{J}^{\mathsf{T}} = \begin{bmatrix} -\frac{2\pi r}{\ell_1}\sin\theta & \frac{2\pi r}{\ell_1}\cos\theta & \frac{L_1}{\ell_1} \\ -\frac{2\pi r}{\ell_2}\sin\theta & \frac{2\pi r}{\ell_2}\cos\theta & \frac{L_2}{\ell_2} \\ \cos\theta & \sin\theta & 0 \end{bmatrix}$$



Unit cell example (a thick tube)

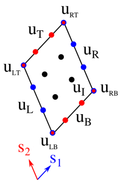
Apply:

1. Displacement boundary conditions of Floquet-Bloch type (two-directional):

$$\mathbf{J}_R^{\mathsf{T}}\mathbf{U}_R = \lambda_1\mathbf{J}_L^{\mathsf{T}}\mathbf{U}_L, \quad \mathbf{J}_T^{\mathsf{T}}\mathbf{U}_T = \lambda_2\mathbf{J}_B^{\mathsf{T}}\mathbf{U}_B, \quad \mathbf{J}_{RB}^{\mathsf{T}}\mathbf{U}_{RB} = \lambda_1\mathbf{J}_{LB}^{\mathsf{T}}\mathbf{U}_{LB}$$
$$\mathbf{J}_{LT}^{\mathsf{T}}\mathbf{U}_{LT} = \lambda_2\mathbf{J}_{LB}^{\mathsf{T}}\mathbf{U}_{LB}, \quad \mathbf{J}_{RT}^{\mathsf{T}}\mathbf{U}_{RT} = \lambda_1\lambda_2\mathbf{J}_{LB}^{\mathsf{T}}\mathbf{U}_{LB}$$

with: $\lambda_1 = \mathrm{e}^{\mathrm{i}k_1\Delta\ell_1}$, $\lambda_2 = \mathrm{e}^{\mathrm{i}k_2\Delta\ell_2}$ (k_1, k_2: helical wavenumbers)

2. Force boundary conditions... (by condensation of displacement)
3. Solve the eigenvalue problem for ω ($k_{1,2}$ are the fixed parameters)



dofs classification (2D view)

Relationship between wavenumbers

- Our structure is of cylindrical type:

$$\begin{cases} k_z = \dfrac{k_2\ell_2 - k_1\ell_1}{L_2 - L_1} \\ k_\theta = \dfrac{k_1\ell_1 L_2 - k_2\ell_2 L_1}{2\pi(L_2 - L_1)} \end{cases}$$

- As opposed to a plate-like geometry, the propagation constants λ_1 and λ_2 are not independent:

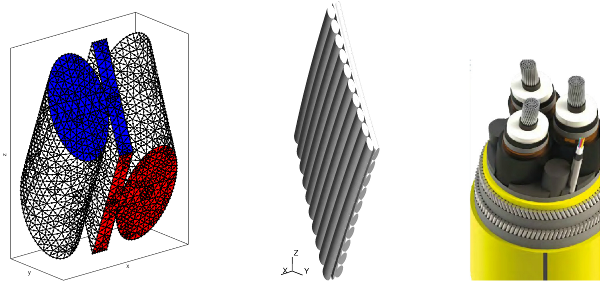
$$\lambda_1^{N_2}\lambda_2^{-N_1} = 1$$

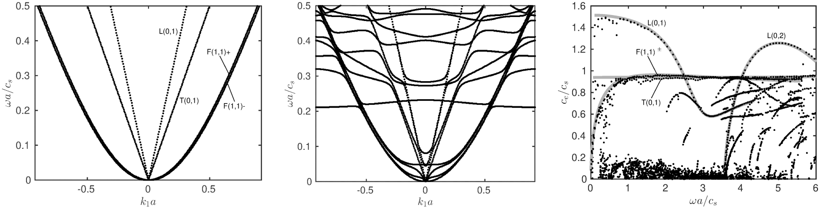
$\Rightarrow$ For (k_1, k_2) in the first Brillouin zone, the circumferential wavenumber is given by:

$$k_\theta = n_\alpha$$

with n_α varying between N_1 consecutive integer values or N_2 consecutive values depending on whether the user sets k_1 or k_2

Application to double armor cables

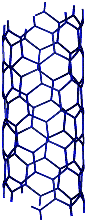


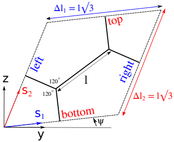


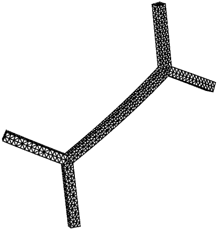
Dispersion curves for: single free wire (left), double armor in low-frequency regime (middle), double armor in high-frequency regime (right, gray: free wire) computed for $n_1 = 0$.

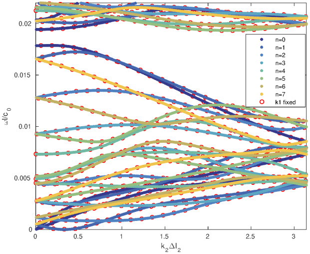
Chiral nanotubes

Test case taken from from Maurin et al., CMAME, 2017







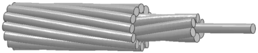


Left: FE mesh of the unit cell. Right: dispersion curves

Conclusion and future works

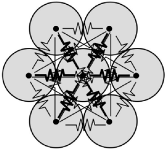
Conclusion and short-term works:

- a modeling framework for Bloch waves in bi-helical structures
- a straightforward extension: beam-like homogenization (quasi-static cable bending fatigue)
- another bi-helical architecture: the double layer strand

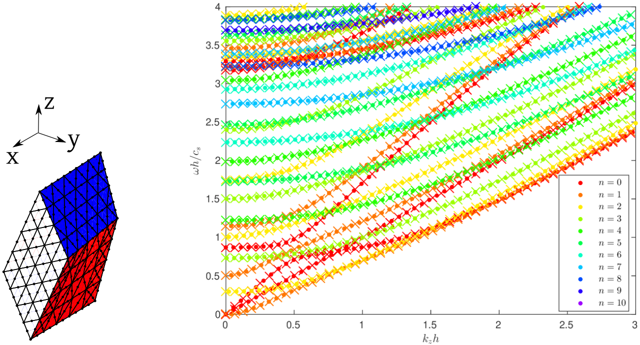


What's next in the mid-term?

- considering more and more heterogeneity...
- work in progress (with Pierric Mora)...
- ... cables simplified as "granular" media (contact theory + beam theory)



Validation for a uniform tube



Thick tube test case ($R/h = 2$). Left: FE mesh of the bi-helical unit cell, right: dispersion curves for various circumferential orders n, computed with the bi-helical unit cell (•), with a commonly used straight unit cell ($\times$) cut along θ and z ($N_\theta = 40$, $\Delta l_z = 0.5h$)

Invariance in arbitrarily curved structures ?

Invariance along s**:** the coefficients of equilibrium equations, including boundary conditions, must be s-periodic...

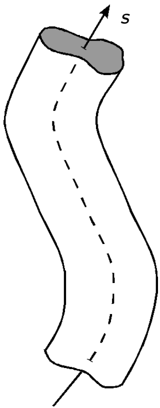
Do guided waves exist in curved structures? Not always, obviously...

Conditions for invariance in a curved direction s**:**

1. cross-section ✓
2. physical properties ✓
3. ?

A third condition is required:

- intuition: difficult...
- inherent to the coordinate system considered (curvilinear)
- answer: **differential geometry and tensor analysis** required...



Elements of differential geometry

Definitions:

- covariant basis: $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = (\partial\mathbf{X}/\partial x, \partial\mathbf{X}/\partial y, \partial\mathbf{X}/\partial s)$
- contravariant basis $\mathbf{g}^i$: $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$
- Christoffel symbols Γ_{ij}^k: $\mathbf{g}_{i,j} = \Gamma_{ij}^k \mathbf{g}_k$ $(\Leftrightarrow \Gamma_{ij}^k = \mathbf{g}_{i,j} \cdot \mathbf{g}^k)$

The coefficients of differential operators are given by the Γ_{ij}^k's

Examples:

$\nabla\varphi = \varphi_{,i}\mathbf{g}^i$

$\nabla\mathbf{u} = (u_{i,j} - \Gamma_{ij}^k u_k)\mathbf{g}^i \otimes \mathbf{g}^j$

$\nabla \cdot \boldsymbol{\sigma} = (\sigma_{ij,j} - \Gamma_{ij}^k \sigma_{kj} - \Gamma_{jj}^k \sigma_{ik})\mathbf{g}^i$

...

The metric tensor

- Definition: $(\mathbf{g})_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$
- **Key formula**: $\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$ $\rightarrow$ functions of the g_{ij}'s only

Metric tensor for usual coordinate systems

cartesian:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

cylindrical:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

spherical:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \varphi \end{bmatrix}$$

Metric tensor for some helical systems

- one-directional helical (x, y, s):
  $\mathbf{X}(x, y, s) = \mathbf{R}(s) + x\mathbf{N}(s) + y\mathbf{B}(s)$
  $$\mathbf{g} = \begin{bmatrix} 1 & 0 & -\tau y \\ 0 & 1 & \tau x \\ -\tau y & \tau x & \tau^2(x^2 + y^2) + (1 - \kappa x)^2 \end{bmatrix}$$
- mixed helical-polar (ρ, θ, s):
  $\mathbf{X}(\rho, \theta, s) = \mathbf{R}(s) + \rho \cos\theta \mathbf{N}(s) + \rho \sin\theta \mathbf{B}(s)$
  $$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & \tau^2 r^2 \\ 0 & \tau^2 r^2 & \tau^2 r^2 + (1 + \kappa r \cos\theta)^2 \end{bmatrix}$$