Effective transmission conditions between fluid and solid domains with non-conform space discretization in transient wave propagation problems

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Presentation of the targeted problem

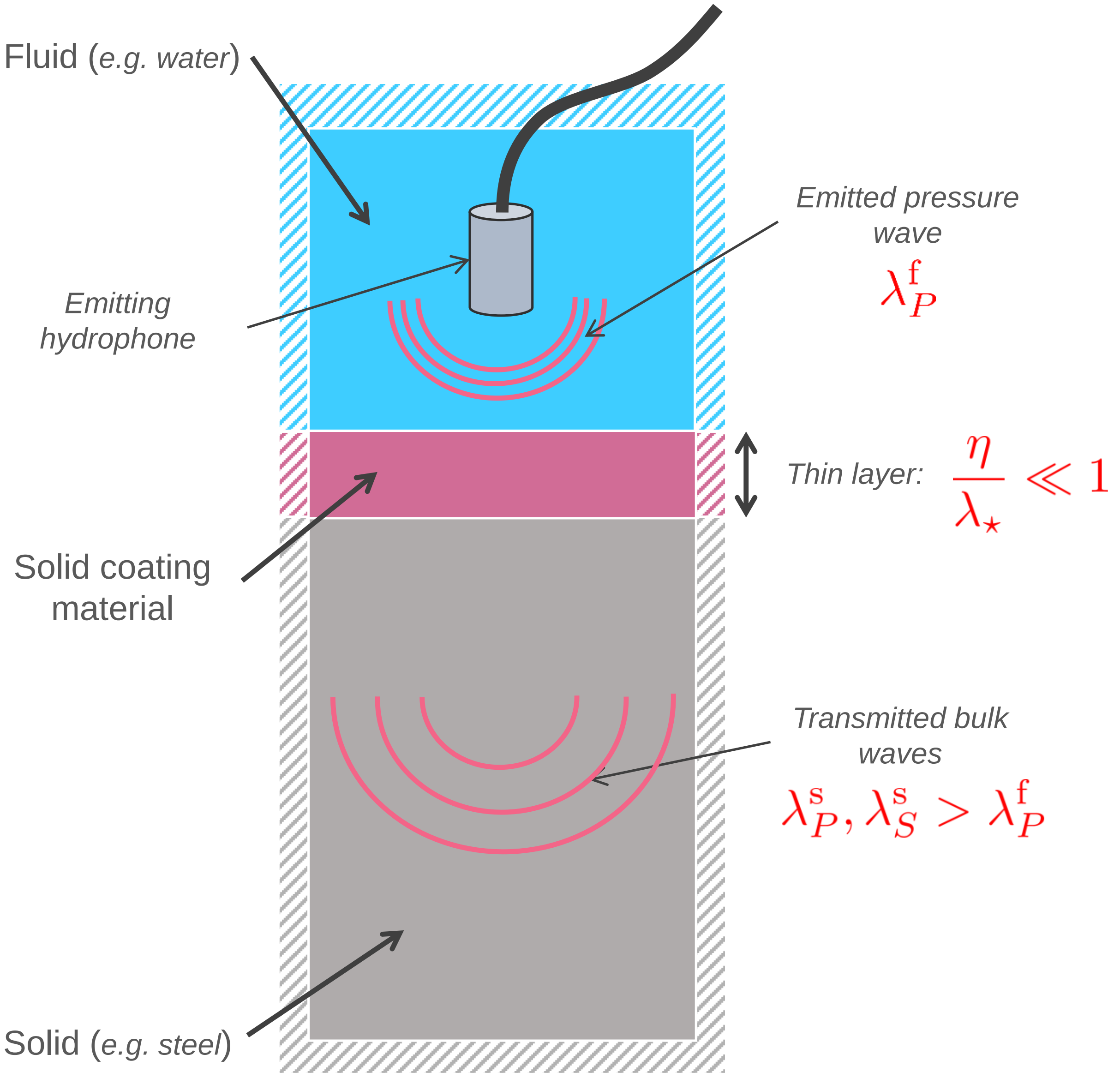
Ultrasonic testing of immersed coated materials, *e.g.*:

- Inspection of radioactive waste drum composed of mortar with a steel coating layer,
- Inspection of metallic pipe repaired with fiber-reinforced composites.

Can we provide an efficient numerical scheme in time-domain for this type of configurations ?

Computational challenges include:

- "Standard" explicit scheme leads to $\Delta t = O(\eta)$
- Conform space discretization leads to **over-refined solid domain,**
- Incorporating **fluid – solid coupling** *(normal stress & velocity continuity)*.



Relevant strategies and proposed approach

- Combining **Discontinuous Galerkin** (leading to noncoformal meshes) with a **local time stepping** approach.

  📖 *J. Diaz, M. Grote, Energy conserving explicit local time stepping for second-order wave equations. SIAM J. Sci. Comput. 2009*

  $\Rightarrow$ **Extension to fluid – solid coupling ?**

- Use asymptotic arguments to derive **Effective Transmission Conditions (ETCs)** between the fluid and the solid domain.

  📖 *S. I. Rokhlin, Y. J. Wang, Analysis of boundary conditions for elastic wave interaction with an interface between two solids, J. Acoust. Soc. Am. 1991*

  📖 *M. Bonnet,* et al. *Effective transmission conditions for thin-layer transmission problems in elastodynamics. The case of a planar layer model. ESAIM: M2AN 2016*

  📖 *N. Lebbe, K. Pham, & A. Maurel, Stable GSTC formulation for Maxwell's equations. IEEE Trans. Antennas Propag. 2022*

  $\Rightarrow$ **Extension to non – conform meshes ? Extension to fluid – solid coupling ?**

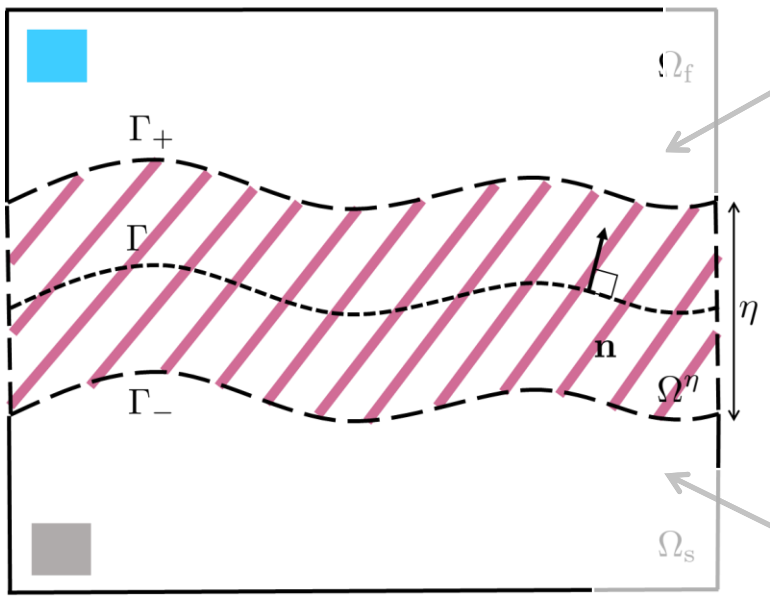
- **Implicit scheme** for the thin layer discrete propagator **with mortar elements**.

  📖 *Chabassier, J., & Imperiale, S. Fourth-order energy-preserving locally implicit time discretization for linear wave equations. Int. J. Numer. Methods Eng. 2016.*

  $\Rightarrow$ **Computational cost associated to an implicit solver ? Extension to fluid – solid coupling ?**

The presented approach **combines the mortar element approach with "*spring – mass*" ETCs**

Recap' fluid – solid coupling $(\eta = 0)$



$$\frac{1}{\rho_\mathrm{f} c^2}\partial^2_{tt}\phi - \frac{1}{\rho_\mathrm{f}}\Delta\phi = f, \qquad \mathbf{v} = \frac{1}{\rho_\mathrm{f}}\nabla\phi, \quad p = -\partial_t\phi, \qquad \text{in} \quad \Omega_\mathrm{f}$$

$\eta = 0$

$$[\mathbf{v}]\cdot\mathbf{n} = 0, \qquad [\boldsymbol{\tau}] = 0, \qquad \text{on} \quad \Gamma$$

Normal velocity continuity — *Normal stress continuity*

$$\rho_\mathrm{s}\partial^2_{tt}\mathbf{u} - \nabla\cdot\boldsymbol{\sigma}(\mathbf{u}) = 0, \qquad \boldsymbol{\sigma}(\mathbf{u}) = \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}), \qquad \text{in} \quad \Omega_\mathrm{s}$$

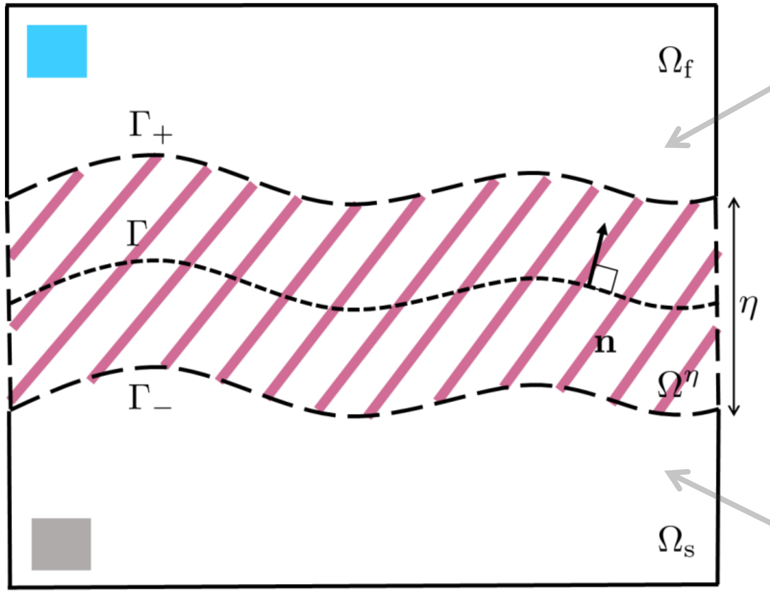
- Continuity conditions written in terms of propagator unknowns:

$$\left|\begin{aligned} 0 = [\mathbf{v}]\cdot\mathbf{n} = \mathbf{v}_+\cdot\mathbf{n} - \mathbf{v}_-\cdot\mathbf{n} &= {}^1/_{\rho_\mathrm{f}}\nabla\phi_+\cdot\mathbf{n} - \partial_t\mathbf{u}_-\cdot\mathbf{n} \\ 0 = [\boldsymbol{\tau}] = \boldsymbol{\tau}_+ - \boldsymbol{\tau}_- &= \partial_t\phi_+\mathbf{n} - \boldsymbol{\sigma}_-\cdot\mathbf{n} \end{aligned}\right.$$

$$\Rightarrow \quad [\mathbf{v}]\cdot\mathbf{n} = 0, \qquad [\boldsymbol{\tau}]\cdot\mathbf{n} = 0, \quad \boldsymbol{\tau}_-\cdot\mathbf{n}^\perp = 0$$

Equivalent form of the continuity conditions

Effective transmission conditions $(\eta > 0)$



$$\frac{1}{\rho_{\rm f} c^2}\partial^2_{tt}\phi - \frac{1}{\rho_{\rm f}}\Delta\phi = f, \qquad \mathbf{v} = \frac{1}{\rho_{\rm f}}\nabla\phi, \quad p = -\partial_t\phi, \qquad \text{in} \quad \Omega_{\rm f}$$

$\eta > 0$

$$[\mathbf{v}]\cdot\mathbf{n} = \eta \mathrm{s}\, \partial_t\langle\boldsymbol{\tau}\rangle\cdot\mathbf{n}, \qquad [\boldsymbol{\tau}]\cdot\mathbf{n} = \eta \mathrm{m}\, \partial_t\langle\mathbf{v}\rangle\cdot\mathbf{n}, \qquad \text{on} \quad \Gamma$$

Constitutive law approximation — *Layer's propagator approximation*

$$\rho_{\rm s}\partial^2_{tt}\mathbf{u} - \nabla\cdot\boldsymbol{\sigma}(\mathbf{u}) = 0, \qquad \boldsymbol{\sigma}(\mathbf{u}) = \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}), \qquad \text{in} \quad \Omega_{\rm s}$$

- The coefficients m and s are **effective mass and compliance** (“spring – mass” ETC),

  📖 *Rohklin, Lombard,* and numerous work on ETCs for wave propagation…

- Other types of ETCs may be considered, *e.g.* extension to **visco-elastic propagators** (*Maxwell, Kelvin-Voigt, Zener, …*)

- This model cannot insure *a priori* convergence. Compared to Bonnet *et al.* we **neglect tangential spatial derivatives**.

  📖 *M. Bonnet,* et al. *Effective transmission conditions for thin-layer transmission problems in elastodynamics. The case of a planar layer model. ESAIM: M2AN 2016*

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Mortar elements and coupled weak formulation

$$\left| \begin{aligned} [\mathbf{v}] \cdot \mathbf{n} &= \eta \mathrm{s}\, \partial_t \langle \boldsymbol{\tau} \rangle \cdot \mathbf{n} \\ [\boldsymbol{\tau}] \cdot \mathbf{n} &= \eta \mathrm{m}\, \partial_t \langle \mathbf{v} \rangle \cdot \mathbf{n} \end{aligned} \right. \quad \overset{\partial_t \lambda_\mathrm{f} = \mathbf{v}_+ \cdot \mathbf{n}, \quad \partial_t \lambda_\mathrm{s} = \boldsymbol{\tau}_- \cdot \mathbf{n}}{\Longrightarrow} \quad \left| \begin{aligned} \lambda_\mathrm{f} - \mathbf{u}_- \cdot \mathbf{n} &= \eta \mathrm{s} \frac{\partial_t \phi_+ + \partial_t \lambda_\mathrm{s}}{2} \\ \phi_+ - \lambda_\mathrm{s} &= \eta \mathrm{m} \frac{\partial_t \lambda_\mathrm{f} + \partial_t \mathbf{u} \cdot \mathbf{n}}{2} \end{aligned} \right.$$

(& time integration)

New variables with **time derivative equal to fluid / solid fluxes** & ETCs are satisfied weakly to **support non-conform meshes**

Ben Belgacem, F., and Y. Maday, The mortar element method for three dimensional finite elements, 1997, and other posterior work…

Find $\{\phi(t), \mathbf{u}(t), \lambda_\mathrm{f}(t), \lambda_\mathrm{s}(t)\} \in H^1(\Omega_\mathrm{f}) \times [H^1(\Omega_\mathrm{s})]^N \times L^2(\Gamma) \times L^2(\Gamma)$, for all $\{\phi^\star, \mathbf{u}^\star, \lambda_\mathrm{f}^\star, \lambda_\mathrm{s}^\star\}$

$$\left| \begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\Omega_\mathrm{f}} \frac{1}{\rho_\mathrm{f} c^2} \phi\, \phi^\star \,\mathrm{d}\Omega + \int_{\Omega_\mathrm{f}} \frac{1}{\rho_\mathrm{f}} \boldsymbol{\nabla} \phi \cdot \boldsymbol{\nabla} \phi^\star \,\mathrm{d}\Omega + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \lambda_\mathrm{f}\, \phi_+^\star \,\mathrm{d}\Gamma = \int_{\Omega_\mathrm{f}} f\, \phi^\star \,\mathrm{d}\Omega, \\ &\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\Omega_\mathrm{s}} \rho_\mathrm{s} \mathbf{u} \cdot \mathbf{u}^\star \,\mathrm{d}\Omega + \int_{\Omega_\mathrm{s}} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u}^\star) \,\mathrm{d}\Omega - \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \lambda_\mathrm{s}\, \mathbf{u}_-^\star \cdot \mathbf{n} \,\mathrm{d}\Gamma = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \eta \mathrm{s} \left(\frac{\phi_+ + \lambda_\mathrm{s}}{2} \right) \lambda_\mathrm{s}^\star \,\mathrm{d}\Gamma = \int_\Gamma (\lambda_\mathrm{f} - \mathbf{u}_- \cdot \mathbf{n}) \lambda_\mathrm{s}^\star \,\mathrm{d}\Gamma, \\ &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \eta \mathrm{m} \left(\frac{\lambda_\mathrm{f} + \mathbf{u}_- \cdot \mathbf{n}}{2} \right) \lambda_\mathrm{f}^\star \,\mathrm{d}\Gamma = \int_\Gamma (\phi_+ - \lambda_\mathrm{s}) \lambda_\mathrm{f}^\star \,\mathrm{d}\Gamma. \end{aligned} \right.$$

- **Second-order dynamics** for the volume propagators,
- Solution satisfies an **energy conservation property**.

Time and space discretization

- In each volume, we use **spectral finite elements with mass lumping,** interface unknowns are discretized using **the trace of solid approximation space.**

$$\mathbb{M}_{\mathrm{f}} \frac{\vec{\phi}^{\,n+1} - 2\vec{\phi}^{\,n} + \vec{\phi}^{\,n+1}}{\Delta t^2} + \mathbb{K}_{\mathrm{f}} \vec{\phi}^{\,n} + \mathbb{C}_{\mathrm{f}} \frac{\vec{\lambda}_{\mathrm{f}}^{n+1} - \vec{\lambda}_{\mathrm{f}}^{n-1}}{2\Delta t} = \vec{f},$$

$$\mathbb{M}_{\mathrm{s}} \frac{\vec{u}^{\,n+1} - 2\vec{u}^{\,n} + \vec{u}^{\,n+1}}{\Delta t^2} + \mathbb{K}_{\mathrm{s}} \vec{u}^{\,n} - \mathbb{C}_{\mathrm{s}} \frac{\vec{\lambda}_{\mathrm{s}}^{n+1} - \vec{\lambda}_{\mathrm{s}}^{n-1}}{2\Delta t} = 0,$$

$\Rightarrow$ **Second order scheme centered at t^n** with explicit stiffness term (leap-frog scheme)

$$\frac{\eta \mathrm{s}}{2} \mathbb{C}_{\mathrm{f}}^{\intercal} \frac{\vec{\phi}^{\,n+1} - \vec{\phi}^{\,n}}{\Delta t} + \frac{\eta \mathrm{s}}{2} \mathbb{M}_{\Gamma} \frac{\vec{\lambda}_{\mathrm{s}}^{n+1} - \vec{\lambda}_{\mathrm{s}}^{n}}{\Delta t} = \mathbb{M}_{\Gamma} \frac{\vec{\lambda}_{\mathrm{f}}^{n+1} + \vec{\lambda}_{\mathrm{f}}^{n}}{2} - \mathbb{C}_{\mathrm{s}}^{\intercal} \frac{\vec{u}^{\,n+1} + \vec{u}^{\,n}}{2},$$

$$\frac{\eta \mathrm{m}}{2} \mathbb{M}_{\Gamma} \frac{\vec{\lambda}_{\mathrm{f}}^{n+1} - \vec{\lambda}_{\mathrm{f}}^{n}}{\Delta t} + \frac{\eta \mathrm{m}}{2} \mathbb{C}_{\mathrm{s}}^{\intercal} \frac{\vec{u}^{\,n+1} - \vec{u}^{\,n}}{\Delta t} = \mathbb{C}_{\mathrm{f}}^{\intercal} \frac{\vec{\phi}^{\,n+1} + \vec{\phi}^{\,n}}{2} - \mathbb{M}_{\Gamma} \frac{\vec{\lambda}_{\mathrm{s}}^{n+1} + \vec{\lambda}_{\mathrm{s}}^{n}}{2}$$

$\Rightarrow$ **Implicit second order scheme centered at $t^{n+1/2}$**

Energy conservation & stability condition

$$\mathbf{K}_{\rm f}^{n+1/2} = \frac{1}{2}\left\|\frac{\vec{\phi}^{\,n+1}-\vec{\phi}^{\,n}}{\Delta t}\right\|^2_{\mathbb{M}_{\rm f}}, \quad \mathbf{P}_{\rm f}^{n+1/2} = \frac{1}{2}\vec{\phi}^{\,n+1\,\intercal}\mathbb{K}_{\rm f}\,\vec{\phi}^{\,n},$$

$$\mathbf{K}_{\rm s}^{n+1/2} = \frac{1}{2}\left\|\frac{\vec{u}^{\,n+1}-\vec{u}^{\,n}}{\Delta t}\right\|^2_{\mathbb{M}_{\rm s}}, \quad \mathbf{P}_{\rm s}^{n+1/2} = \frac{1}{2}\vec{u}^{\,n+1\,\intercal}\mathbb{K}_{\rm s}\,\vec{u}^{\,n}$$

$\Rightarrow$ **Kinetic & potential energy** of the volume propagators

$$\mathbf{K}_{\eta}^{n+1/2} = \frac{\eta \mathrm{m}}{2}\left\|\frac{1}{2}\frac{\vec{\lambda}_{\rm f}^{\,n+1}-\vec{\lambda}_{\rm f}^{\,n}}{\Delta t} + \frac{1}{2}\mathbb{M}_{\Gamma}^{-1}\mathbb{C}_{\rm s}^{\intercal}\frac{\vec{u}^{\,n+1}-\vec{u}^{\,n}}{\Delta t}\right\|^2_{\mathbb{M}_{\Gamma}},$$

$$\mathbf{P}_{\eta}^{n+1/2} = \frac{\eta \mathrm{s}}{2}\left\|\frac{1}{2}\frac{\vec{\lambda}_{\rm s}^{\,n+1}-\vec{\lambda}_{\rm s}^{\,n}}{\Delta t} + \frac{1}{2}\mathbb{M}_{\Gamma}^{-1}\mathbb{C}_{\rm f}^{\intercal}\frac{\vec{\phi}^{\,n+1}-\vec{\phi}^{\,n}}{\Delta t}\right\|^2_{\mathbb{M}_{\Gamma}}.$$

$\Rightarrow$ **Kinetic & potential energy** of the interface

The total energy satisfies a discrete energy conservation property

$$\frac{\mathbf{E}^{n+1/2}-\mathbf{E}^{n-1/2}}{\Delta t} = 0, \qquad \mathbf{E}^{n+1/2} = \mathbf{K}_{\rm f}^{n+1/2} + \mathbf{P}_{\rm f}^{n+1/2} + \mathbf{K}_{\rm s}^{n+1/2} + \mathbf{P}_{\rm s}^{n+1/2} + \mathbf{K}_{\eta}^{n+1/2} + \mathbf{P}_{\eta}^{n+1/2}$$

One can verify that the CFL condition insuring positivity of the total energy functional **does not depend on the thickness & effective parameters**

$$\Delta t \le \min\left\{\frac{2}{\sqrt{r(\mathbb{M}_{\rm f}^{-1}\mathbb{K}_{\rm f})}}, \frac{2}{\sqrt{r(\mathbb{M}_{\rm s}^{-1}\mathbb{K}_{\rm s})}}\right\}, \qquad r(\cdot) \text{ matrix spectral radius}$$

Time marching algorithm

Fully coupled discrete system $\vec{X}^n = \begin{pmatrix} \vec{\phi}^n & \vec{u}^n \end{pmatrix}^\intercal, \quad \vec{L}^n = \begin{pmatrix} \vec{\lambda}_\mathrm{f}^n & \vec{\lambda}_\mathrm{s}^n \end{pmatrix}^\intercal \quad \Rightarrow \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C}^\intercal & \mathbb{D} \end{pmatrix} \begin{pmatrix} \vec{X}^{n+1} \\ \vec{L}^{n+1} \end{pmatrix} = \begin{pmatrix} \vec{F} + \mathbb{B}\vec{L}^{n-1} \\ \vec{G} \end{pmatrix}$

$$\mathbb{A} = \begin{pmatrix} \mathbb{M}_\mathrm{f} & \\ & \mathbb{M}_\mathrm{s} \end{pmatrix}, \ \mathbb{B} = \frac{\Delta t}{2} \begin{pmatrix} \mathbb{C}_\mathrm{f} & \\ & -\mathbb{C}_\mathrm{s} \end{pmatrix}, \ \mathbb{C}^\intercal = \frac{1}{2} \begin{pmatrix} -\mathbb{C}_\mathrm{f}^\intercal & \frac{\eta \mathrm{m}}{\Delta t} \mathbb{C}_\mathrm{s}^\intercal \\ \frac{\eta \mathrm{s}}{\Delta t} \mathbb{C}_\mathrm{f}^\intercal & \mathbb{C}_\mathrm{s}^\intercal \end{pmatrix}, \ \mathbb{D} = \frac{1}{2} \begin{pmatrix} \frac{\eta \mathrm{m}}{\Delta t} \mathbb{M}_\Gamma & \mathbb{M}_\Gamma \\ -\mathbb{M}_\Gamma & \frac{\eta \mathrm{s}}{\Delta t} \mathbb{M}_\Gamma \end{pmatrix}$$

Diagonal matrix thanks to mass lumping

Introducing the **invertible** Schur complement matrix $\mathbb{S} = \mathbb{D} - \mathbb{C}^\intercal \mathbb{A}^{-1} \mathbb{B}$ the solution is obtained through

$$\left| \begin{array}{l} \vec{L}^{n+1} = \mathbb{S}^{-1}(\vec{G} - \mathbb{C}^\intercal \mathbb{A}^{-1}(\vec{F} + \mathbb{B}\vec{L}^{n-1})) \\ \vec{X}^{n+1} = \mathbb{A}^{-1}(\vec{F} + \mathbb{B}(\vec{L}^{n-1} - \vec{L}^{n+1})) \end{array} \right.$$

- **In the conform case,** the Schur complement can be **lumped (block diagonal),**
- In the non-conform case: a linear system needs to be solved at each time step.

In practice we implement the "**weakly invasive** coupling algorithm" :

1 – **Propagators pre-process step:** $\vec{X}^* \leftarrow \vec{F}$,

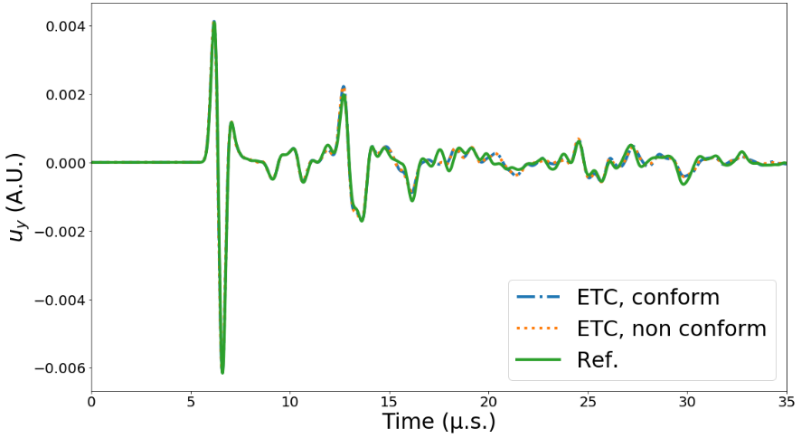
2 – **Interface pre-process step:** $\vec{X}^* \leftarrow \vec{X}^* + \mathbb{B}\vec{L}^{n-1}$,

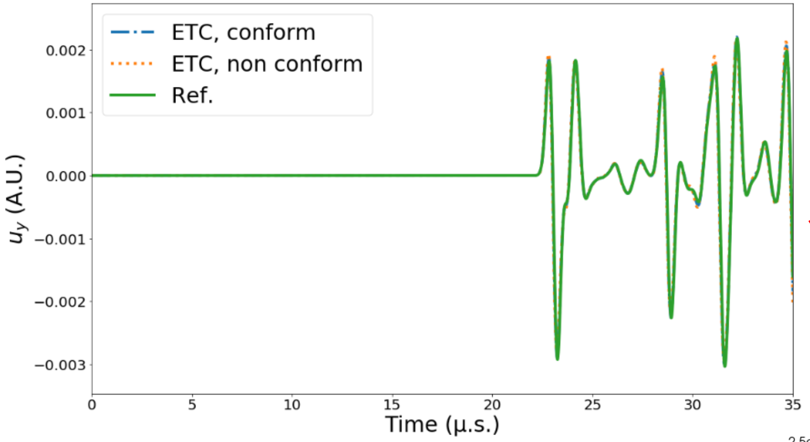
3 – **Propagators compute step:** $\vec{X}^* \leftarrow \mathbb{A}^{-1}\vec{X}^*$,

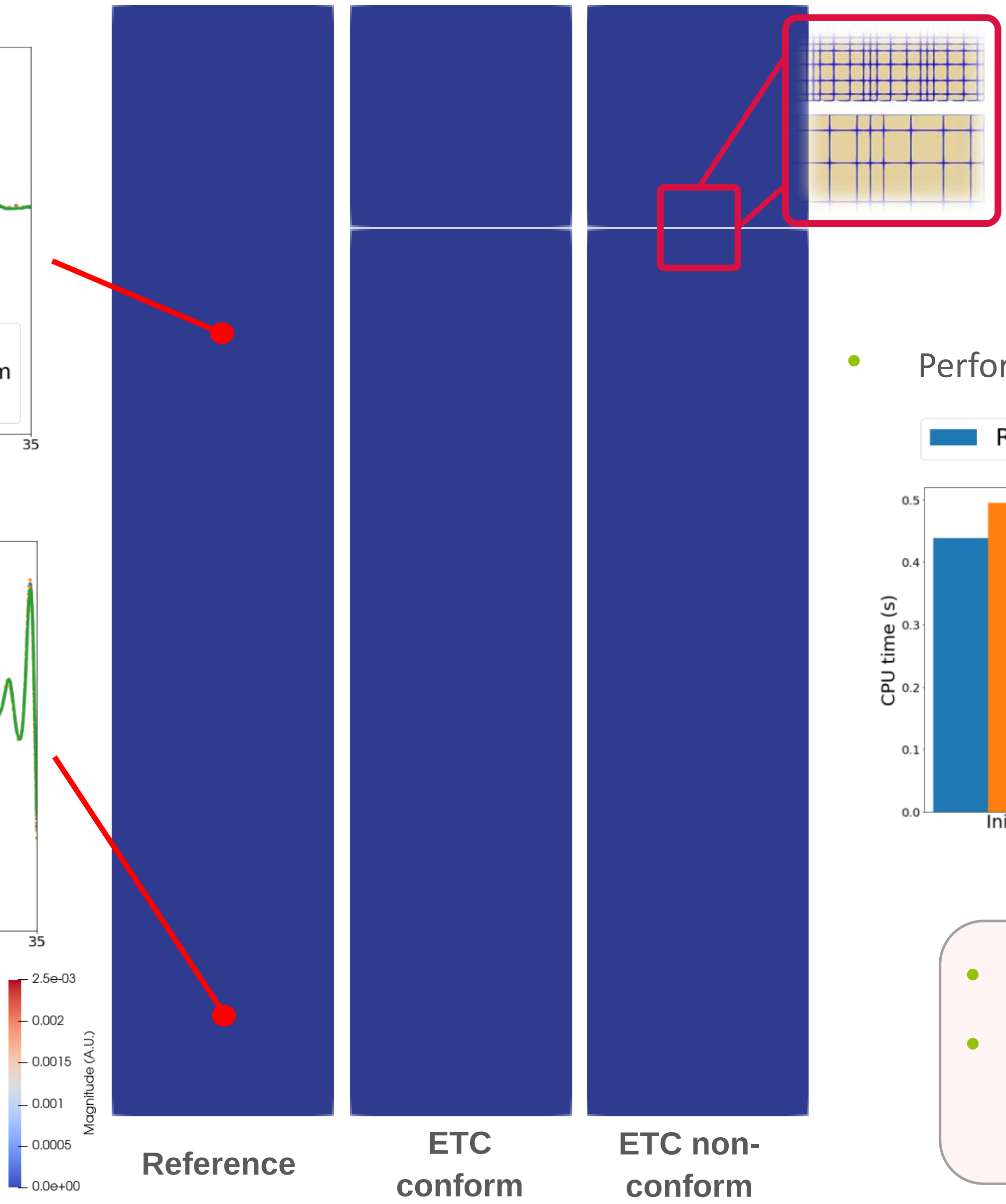
4 – **Interface compute step:** $\vec{L}^{n+1} \leftarrow \mathbb{S}^{-1}(\vec{G} - \mathbb{C}^\intercal \vec{X}^*)$,

5 – **Propagators post-process step:** $\vec{X}^{n+1} \leftarrow \vec{X}^* - \mathbb{A}^{-1}\mathbb{B}\vec{L}^{n+1}$.

Results 2D

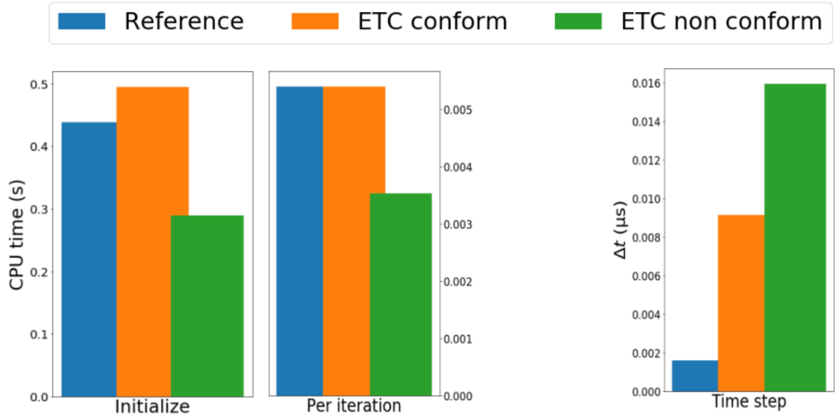






- Configuration parameters:
  - 50 (longitudinal) wavelengths depth in fluid and solid domains,
  - Coating material twice as stiff as solid domain,
  - $\eta = \lambda_L / 10$

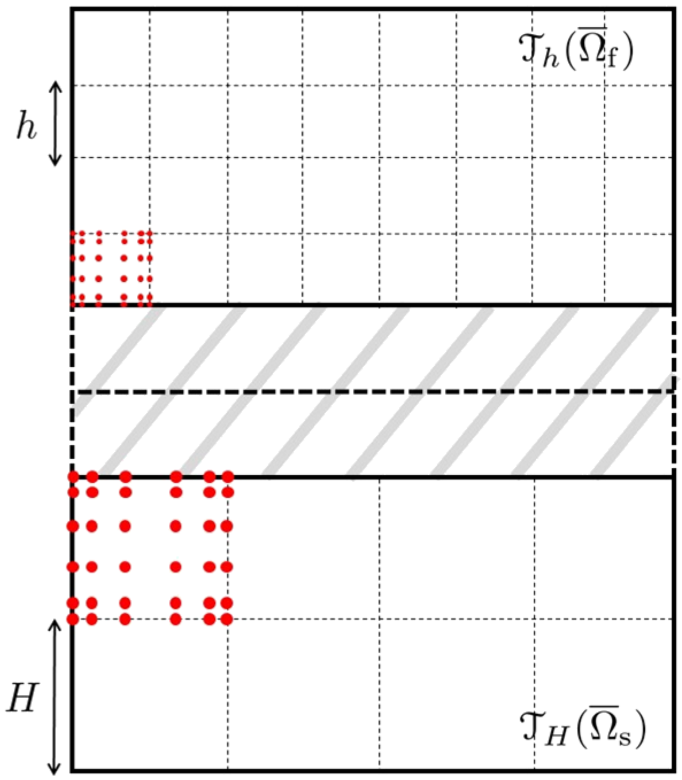
- Performance improvements:



÷ 1.52 **x 9.96**

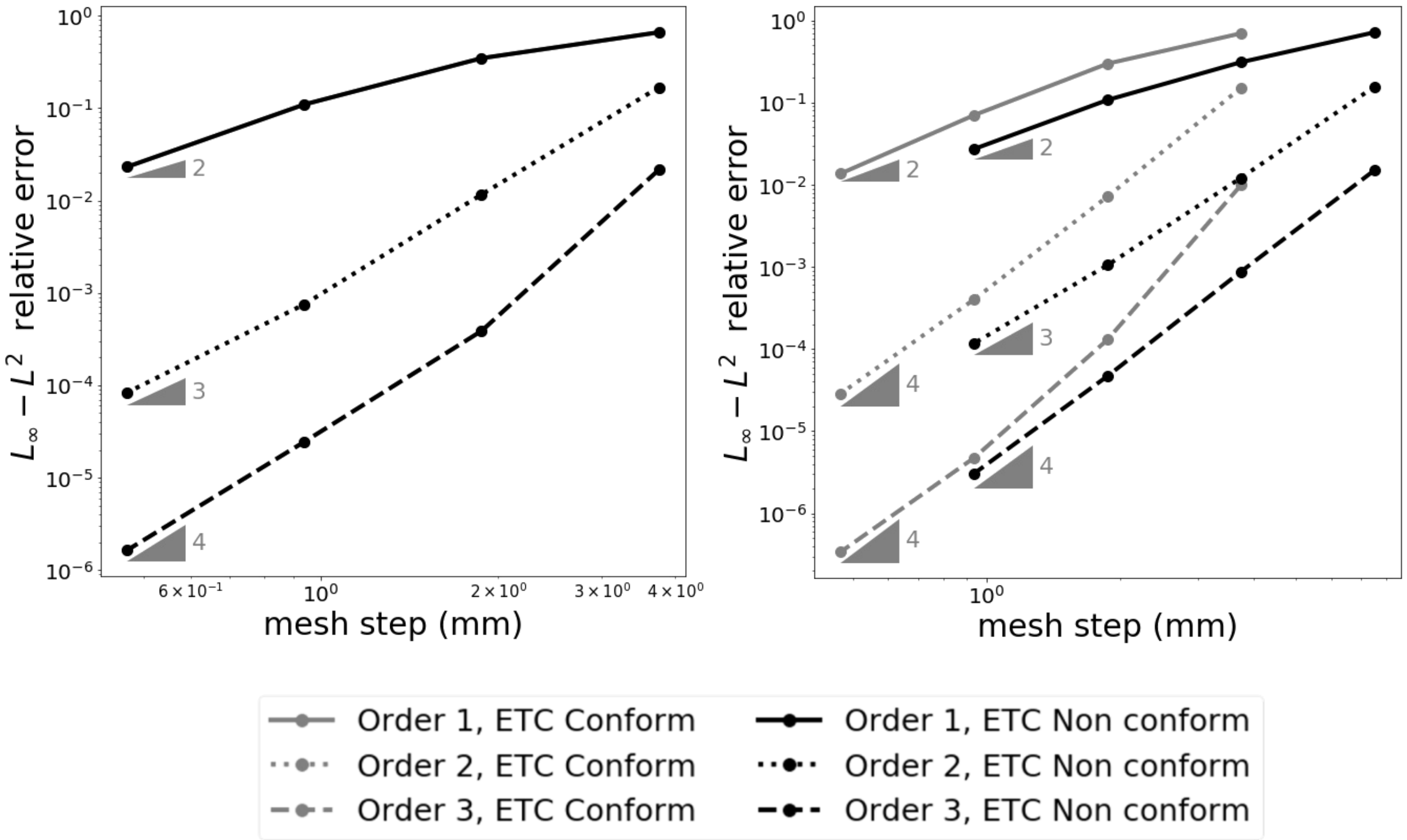
- x15 speed up,
- Very good agreement for volume waves, limited discrepancies on surface waves due to ETCs.

Illustration of high-order space convergence

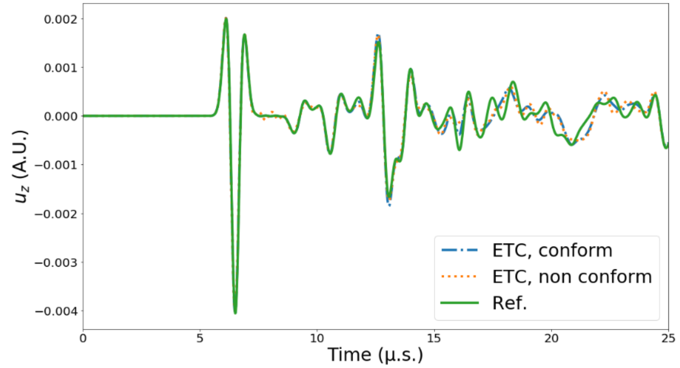


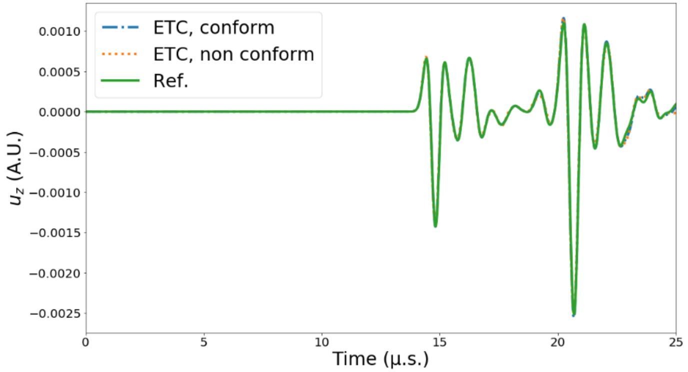
The configuration used to numerically test space convergence

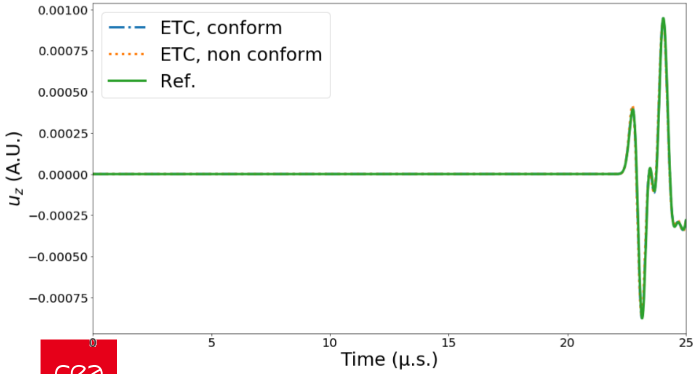
- Does computation of "$L^\infty(L^2)$" shows expected convergence in $O(\Delta t^2 + h^{p+1})$
- Time step is identical to the reference solution to **highlight space discretization errors.**

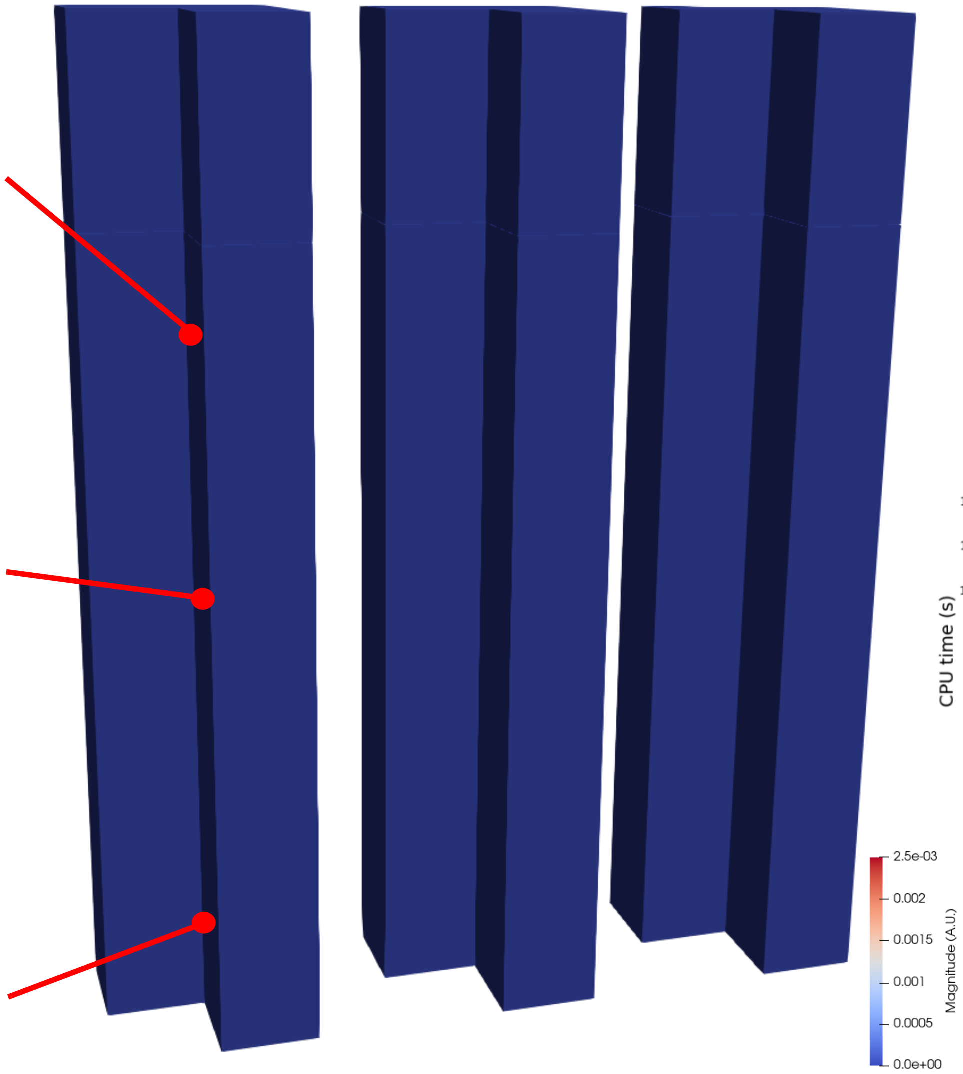


Results 3D







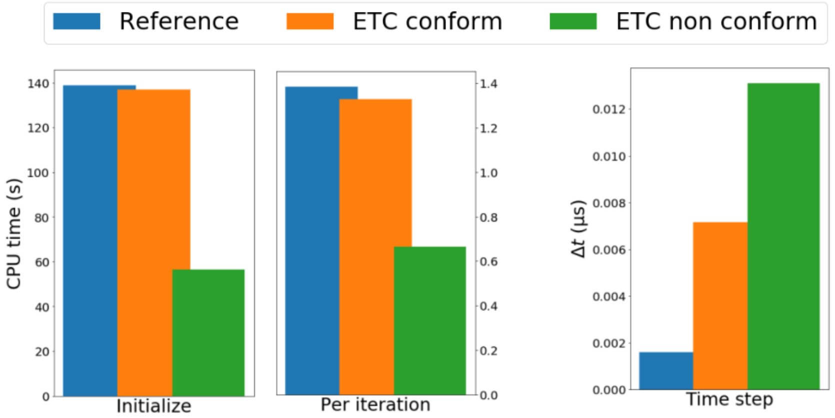


Reference 165 x 10^6 DoFs

ETC conform 163 x 10^6 DoFs

ETC non-conform 61 x 10^6 DoFs

- Configuration parameters:
  - 50 (longitudinal) wavelengths depth in fluid and solid domains,
  - Coating material twice as stiff as solid domain,
  - $\eta = \lambda_L/10$
- Performance improvements:



÷ 2.07 **x 8.30**

- x17 speed up,
- Very good agreement for volume waves, limited discrepancies on surface waves due to ETCs.

Wrap up' & perspectives

Wrap up' !

- Combining **the mortar element method** with a **"spring-mass" ETC** between **fluid – solid domains**.

$\Rightarrow$ Incorporating **spatial non-conformities.**

$\Rightarrow$ CFL condition independent of coating layer **thickness** and **effective parameters.**

$\Rightarrow$ **Significant performance gain** for simple 2D/3D cases

Potential perspectives…

- Use of more **elaborated ETCs** (e.g. obtained from asymptotic analysis),
- Use this robustness w.r.t. thickness / effective parameters within **inversion loops**

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