Simulation of homogenized subwavelength metasurfaces

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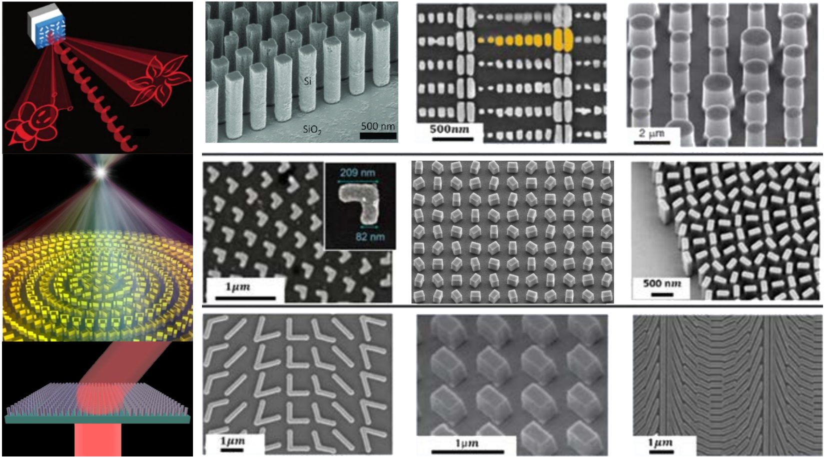
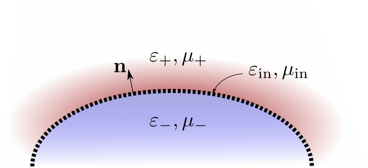
Metasurfaces in nanophotonics



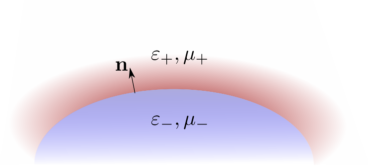
Electromagnetic transition conditions

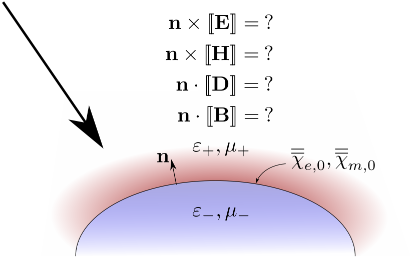
$$\mathbf{n} \times [\![\mathbf{E}]\!] = 0$$
$$\mathbf{n} \times [\![\mathbf{H}]\!] = 0$$
$$\mathbf{n} \cdot [\![\mathbf{D}]\!] = 0$$
$$\mathbf{n} \cdot [\![\mathbf{B}]\!] = 0$$



$$\nabla \times \mathbf{E} = -i\omega\mathbf{B}$$
$$\nabla \times \mathbf{H} = i\omega\mathbf{D}$$
$$\nabla \cdot \mathbf{D} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{n} \times [\![\mathbf{E}]\!] = ?$$
$$\mathbf{n} \times [\![\mathbf{H}]\!] = ?$$
$$\mathbf{n} \cdot [\![\mathbf{D}]\!] = ?$$
$$\mathbf{n} \cdot [\![\mathbf{B}]\!] = ?$$





Surfacic material properties

We assume than the susceptibilities in the whole domain are given by :

$$\overline{\overline{\chi}}_e = \chi_e^{\pm} + \overline{\overline{\chi}}_{e,0}\delta_S \quad \text{with} \quad \chi_e^{\pm} = \begin{cases} \varepsilon_+ - 1 & \text{in} \quad \mathcal{D}_+ \\ \varepsilon_- - 1 & \text{in} \quad \mathcal{D}_- \end{cases}$$

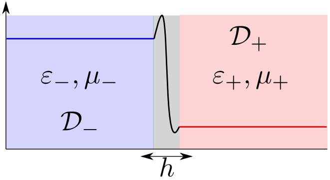
with δ_S the **Dirac distribution on the interface** S.

You can see this decomposition as the limit of the **first order expansion** of the susceptibilities when the **thickness of the metasurface tends to zero**.

This also means that :

$$\left[\overline{\overline{\chi}}_{e,0}\right] = \mathrm{m} \quad \text{and} \quad \overline{\overline{\chi}}_{e,0} \propto h$$

where h is the thickness of the metasurface.



GSTC Derivation

Finding the jump conditions verified by the fields is achieved by injecting the decompositions into Maxwell's equations :

$$\begin{aligned} \nabla \times \mathbf{E} &= -i\omega \mathbf{B}, \\ \nabla \times \mathbf{H} &= i\omega \mathbf{D}, \\ \nabla \cdot \mathbf{D} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \quad \xleftarrow{\text{injection}} \quad \begin{aligned} &\overline{\overline{\chi}}_e = \chi_e^{\pm} + \overline{\overline{\chi}}_{e,0}\delta_S \\ &\mathbf{A} = \mathbf{A}^{\pm} + \mathbf{A}_0\delta_S \quad \text{for} \quad \mathbf{A} = \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B} \\ &\left.\begin{aligned} \mathbf{D} &= (\overline{\overline{\chi}}_e + 1)\mathbf{E} \\ \mathbf{B} &= (\overline{\overline{\chi}}_m + 1)\mathbf{H} \end{aligned}\right\} \text{Constitutive relations (anisotropic, linear)} \end{aligned}$$

This lead to the following Generalized Sheet Transition Conditions :

$$\mathbf{n} \times [\![\mathbf{E}]\!] = \nabla_{\parallel} \times \left(\overline{\overline{\chi}}_{e,0}\left\{\mathbf{E}\right\}\right)_{\perp} - i\omega\mu_0 \left(\overline{\overline{\chi}}_{m,0}\left\{\mathbf{H}\right\}\right)_{\parallel},$$

$$\mathbf{n} \times [\![\mathbf{H}]\!] = \nabla_{\parallel} \times \left(\overline{\overline{\chi}}_{m,0}\left\{\mathbf{H}\right\}\right)_{\perp} + i\omega\varepsilon_0 \left(\overline{\overline{\chi}}_{e,0}\left\{\mathbf{E}\right\}\right)_{\parallel},$$

$$\mathbf{n} \cdot [\![\mathbf{D}]\!] = -\varepsilon_0 \nabla_{\parallel} \cdot \left(\overline{\overline{\chi}}_{e,0}\left\{\mathbf{E}\right\}\right)_{\parallel},$$

$$\mathbf{n} \cdot [\![\mathbf{B}]\!] = -\mu_0 \nabla_{\parallel} \cdot \left(\overline{\overline{\chi}}_{m,0}\left\{\mathbf{H}\right\}\right)_{\parallel}.$$

Susceptibility synthesis

Several physicists considered these transition conditions as a way to **synthesize** new metasurfaces.

- Using some "physical intuitions", we can assume that the susceptibilities are of the following form :

$$\overline{\overline{\chi}}_{e,0} = \begin{pmatrix} \chi_{e,0}^{xx} & 0 & 0 \\ 0 & \chi_{e,0}^{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \overline{\overline{\chi}}_{m,0} = \begin{pmatrix} \chi_{m,0}^{xx} & 0 & 0 \\ 0 & \chi_{m,0}^{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- With this assumption we have :

$$\mathbf{n} \times [\![\mathbf{E}]\!] = -i\omega\mu_0 \left(\overline{\overline{\chi}}_{m,0} \{\mathbf{H}\}\right)_{\|} \quad \text{and} \quad \mathbf{n} \times [\![\mathbf{H}]\!] = +i\omega\varepsilon_0 \left(\overline{\overline{\chi}}_{e,0} \{\mathbf{E}\}\right)_{\|} .$$

- If we constrain the fields above and below the metasurface, we have :

$$\chi_{m,0}^{xx} = \frac{\mathbf{n} \times [\![\mathbf{E}]\!] \cdot \mathbf{x}}{-i\omega\mu_0 \{\mathbf{H}\} \cdot \mathbf{x}} \quad \text{etc.}$$

Inversion method : deflector

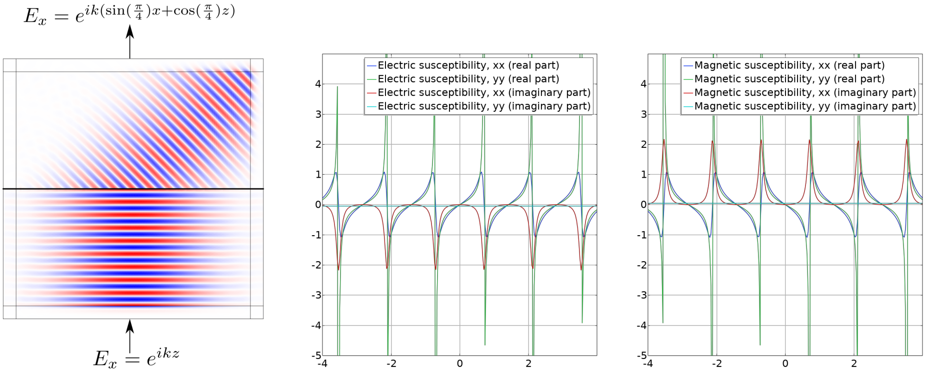


Fig. Deflecting a normal incident plane wave by $\frac{\pi}{4}$.

Inversion method : lens

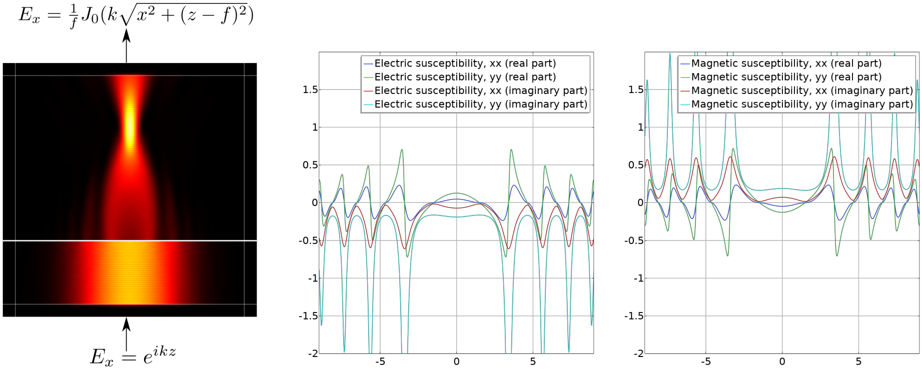


Fig. “Perfect” lens making normal incident plane wave converge at a focal point.

Inversion method : cloaking

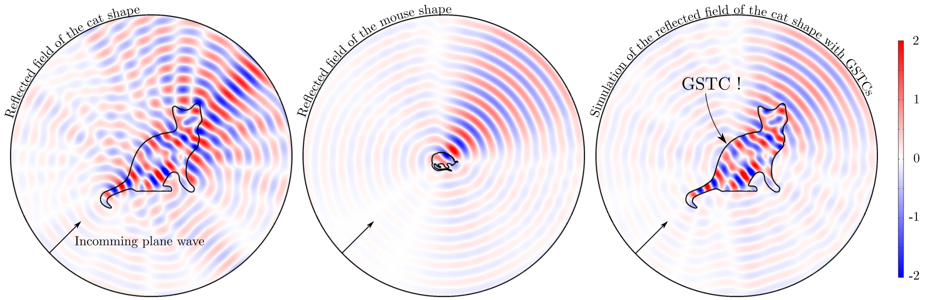
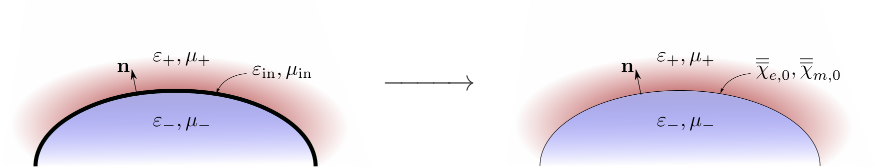


Fig. Cloaking system where susceptibilities are synthesized in such a way that the reflected field obtained when injecting a plane wave on the "cat" is equal to the one reflected by the "mouse".

Thin homogeneous layer

Before considering the microstructuration of a metasurface, let us have a look to the case of thin ($h \ll \lambda$) homogeneous layers :

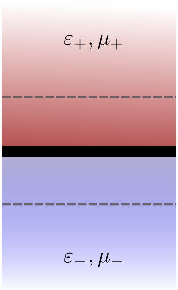


The equivalent transmission conditions are obtained through an **asymptotic expansion** of the near and far fields when $h \to 0$.

Asymptotic expansion

We consider the following expansions :

$$\text{(Near field)}\ \mathbf{A}(\mathbf{x}) = \sum_{n=0}^{\infty} h^n \mathbf{a}_n(\mathbf{x}_\parallel, \xi_\perp)$$



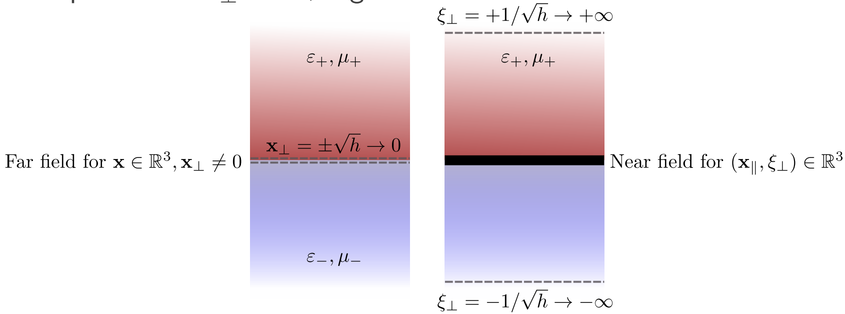
$$\mathbf{A}(\mathbf{x}) = \sum_{n=0}^{\infty} h^n \mathbf{A}_n(\mathbf{x})\ \text{(Far field)}$$

The "slowly"-varying variables **x** are used to find the macroscopic behavior of the fields while the microscopic (or "rapidly"-varying) variables $\boldsymbol{\xi} = \mathbf{x}/h$ are usefull to describe the near field interactions.

(!) No further hypothesis are made on the material properties !

Matching conditions

We need some conditions to link the values of the near and far fields. A Taylor expansion at $\mathbf{x}_\perp = \pm\sqrt{h}$ gives :



This lead to the following **matching conditions** :

$$[\![\mathbf{A}_0]\!] = \lim_{\xi_\perp \to +\infty} \mathbf{a}_0(\mathbf{x}_\parallel, \xi_\perp) - \mathbf{a}_0(\mathbf{x}_\parallel, -\xi_\perp) = 0 \quad (\,!\,),$$

$$[\![\mathbf{A}_1]\!] = \lim_{\xi_\perp \to +\infty} \mathbf{a}_1(\mathbf{x}_\parallel, \xi_\perp) - \mathbf{a}_1(\mathbf{x}_\parallel, -\xi_\perp) - 2\xi_\perp \nabla_\perp \{\mathbf{A}_0\}.$$

Thin homogeneous layer susceptibilities

If the near fields are known, the transmission conditions verified by the macroscopic fields are found using $[\![\mathbf{A}]\!] \simeq [\![\mathbf{A}_0]\!] + h\,[\![\mathbf{A}_1]\!]$.

In this case, the near fields are given analytically and we find the following transmission conditions :

$$
\begin{aligned}
\mathbf{n} \times [\![\mathbf{E}]\!] &= -i\omega\mu_0(\mu_{\mathrm{in}} - 1)h\left\{\mathbf{H}_\parallel\right\}, \\
\mathbf{n} \times [\![\mathbf{H}]\!] &= i\omega\varepsilon_0(\varepsilon_{\mathrm{in}} - 1)h\left\{\mathbf{E}_\parallel\right\}, \\
\mathbf{n} \cdot [\![\mathbf{D}]\!] &= -\varepsilon_0(\varepsilon_{\mathrm{in}} - 1)h\nabla_\parallel \cdot \left\{\mathbf{E}_\parallel\right\}, \\
\mathbf{n} \cdot [\![\mathbf{B}]\!] &= -\mu_0(\mu_{\mathrm{in}} - 1)h\nabla_\parallel \cdot \left\{\mathbf{H}_\parallel\right\}.
\end{aligned}
$$

$$
\overline{\overline{\chi}}_{e,0} = h\begin{pmatrix} \chi_{e,\mathrm{in}} & 0 & 0 \\ 0 & \chi_{e,\mathrm{in}} & 0 \\ 0 & 0 & 0 \end{pmatrix}
\qquad
\overline{\overline{\chi}}_{m,0} = h\begin{pmatrix} \chi_{m,\mathrm{in}} & 0 & 0 \\ 0 & \chi_{m,\mathrm{in}} & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

Enlarged interface

Instead of jump conditions between $z = \pm 0$, we can consider the real thickness of the microstructure and use transition conditions between $\pm h/2$.

With a Taylor expansion at $\pm h/2$ we get from the GSTCs :

$$\mathbf{A}(x, y, \pm h/2) = \mathbf{A}(x, y, \pm 0) \pm h/2 \nabla_\perp \mathbf{A}(x, y, \pm 0) + o(h)$$

$$\Rightarrow \quad [\![\mathbf{A}]\!]_{\pm h/2} = [\![\mathbf{A}]\!] + h \{\nabla_\perp \mathbf{A}\} + o(h) \quad \text{and} \quad \{\mathbf{A}\}_{\pm h/2} = \{\mathbf{A}\} + o(h).$$

$$\mathbf{n} \times [\![\mathbf{E}]\!] = \nabla_\| \times \left(\overline{\overline{\chi}}_{e,0} \{\mathbf{E}\}\right)_\perp - i\omega\mu_0 \left(\overline{\overline{\chi}}_{m,0} \{\mathbf{H}\}\right)_\| ,$$
$$\mathbf{n} \times [\![\mathbf{H}]\!] = \nabla_\| \times \left(\overline{\overline{\chi}}_{m,0} \{\mathbf{H}\}\right)_\perp + i\omega\varepsilon_0 \left(\overline{\overline{\chi}}_{e,0} \{\mathbf{E}\}\right)_\|$$

The susceptibilities are changed :

$$\overline{\overline{\chi}}_{e,0} \to \overline{\overline{\chi}}_{e,0} + h \begin{pmatrix} \varepsilon_r & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Example in 2D

$$-\nabla \cdot a\nabla u - k^2 bu = 0$$
$$[\![u]\!] = \{a_0^{yy}\partial_y u\}$$
$$[\![\partial_y u]\!] = k^2\{b_0 u\} - \{\partial_x a_0^{xx}\partial_x u\}$$

$$b_{0,\pm} = h\left(\frac{b_- + b_+}{2} - b_{\text{in}} - b_\pm\right)$$
$$a_{0,\pm}^{xx} = h\left(a_{\text{in}} - \frac{a_- + a_+}{2} + a_\pm\right)$$
$$a_{0,\pm}^{yy} = h\left(\frac{1}{a_{\text{in}}} - \frac{\frac{1}{a_-} + \frac{1}{a_+}}{2} + \frac{1}{a_\pm}\right)$$

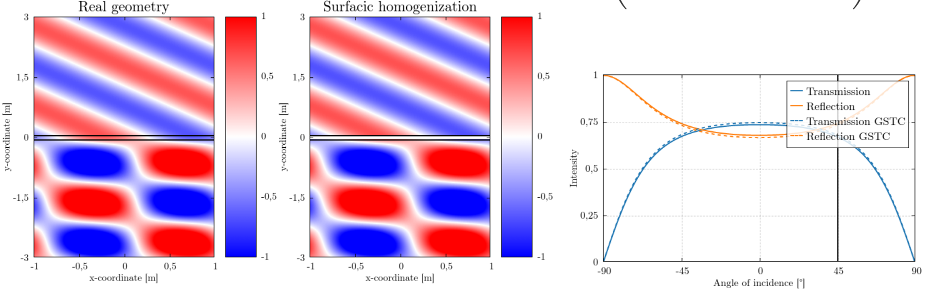


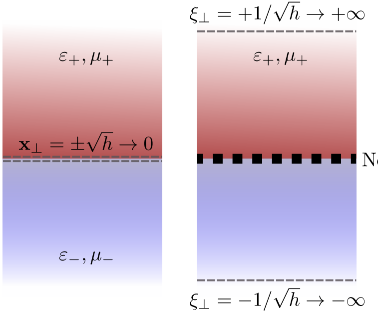
Fig. Example : $a_\pm = b_\pm = 1$, $a_{\text{in}} = 2$, $b_{\text{in}} = 5$, $h = 0.1$ and $k = 2\pi/\lambda$ with $\lambda = 1$.

Surfacic homogenization

For periodic metasurfaces, since the period of the inclusions is proportional to the thickness of the structure, we need to use both an asymptotic expansion and homogenization for the tangential components.

Far field for $\mathbf{x} \in \mathbb{R}^3, \mathbf{x}_\perp \neq 0$

$$\mathbf{A}(\mathbf{x}) = \sum_{n=0}^{\infty} h^n \mathbf{A}_n(\mathbf{x})$$



Near field for $(\mathbf{x}_\parallel, \xi_\perp) \in \mathbb{R}^3$

$$\mathbf{A}(\mathbf{x}) = \sum_{n=0}^{\infty} h^n \mathbf{a}_n(\mathbf{x}_\parallel, \boldsymbol{\xi})$$

$$\boldsymbol{\xi} = \frac{\mathbf{x}}{h}$$

This time the near fields can't be solved analytically !

Elementary problems

Injecting the series expansion of the fields into Maxwell equations we find that the near fields are given by a basis of "elementary problems" $\boldsymbol{\mathcal{E}}_i$:

$$\mathbf{e}_0(\mathbf{x}_\parallel, \xi) = \boldsymbol{\mathcal{E}}_x(\xi)\mathbf{x} \cdot \mathbf{E}_0(\mathbf{x}_\parallel, 0) + \boldsymbol{\mathcal{E}}_y(\xi)\mathbf{y} \cdot \mathbf{E}_0(\mathbf{x}_\parallel, 0) + \boldsymbol{\mathcal{E}}_z(\xi)\mathbf{z} \cdot \mathbf{E}_0(\mathbf{x}_\parallel, 0)$$

with

$$\begin{aligned} \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\mathcal{E}}_x &= 0, \\ \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\mathcal{D}}_x &= 0, \\ \boldsymbol{\mathcal{D}}_x &= \varepsilon_0 \varepsilon_r \boldsymbol{\mathcal{E}}_x, \\ \mathbf{x} \cdot \boldsymbol{\mathcal{E}}_x(\xi_\parallel, \pm\infty) &= 1 \\ \mathbf{y} \cdot \boldsymbol{\mathcal{E}}_x(\xi_\parallel, \pm\infty) &= 0 \\ \mathbf{z} \cdot \boldsymbol{\mathcal{E}}_x(\xi_\parallel, \pm\infty) &= 0 \end{aligned} \qquad \begin{aligned} \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\mathcal{E}}_y &= 0, \\ \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\mathcal{D}}_y &= 0, \\ \boldsymbol{\mathcal{D}}_y &= \varepsilon_0 \varepsilon_r \boldsymbol{\mathcal{E}}_y, \\ \mathbf{x} \cdot \boldsymbol{\mathcal{E}}_y(\xi_\parallel, \pm\infty) &= 0 \\ \mathbf{y} \cdot \boldsymbol{\mathcal{E}}_y(\xi_\parallel, \pm\infty) &= 1 \\ \mathbf{z} \cdot \boldsymbol{\mathcal{E}}_y(\xi_\parallel, \pm\infty) &= 0 \end{aligned} \qquad \begin{aligned} \nabla_{\boldsymbol{\xi}} \times \boldsymbol{\mathcal{E}}_z &= 0, \\ \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\mathcal{D}}_z &= 0, \\ \boldsymbol{\mathcal{D}}_z &= \varepsilon_0 \varepsilon_r \boldsymbol{\mathcal{E}}_z, \\ \mathbf{x} \cdot \boldsymbol{\mathcal{E}}_z(\xi_\parallel, \pm\infty) &= 0 \\ \mathbf{y} \cdot \boldsymbol{\mathcal{E}}_z(\xi_\parallel, \pm\infty) &= 0 \\ \mathbf{z} \cdot \boldsymbol{\mathcal{E}}_z(\xi_\parallel, \pm\infty) &= 1 \end{aligned}$$

Susceptibilities

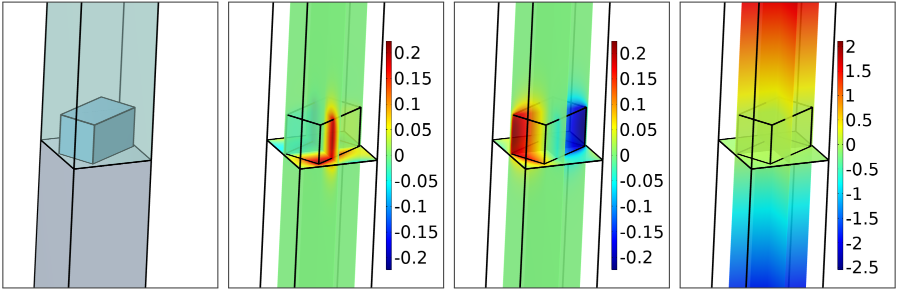


Fig. Example of elementary problems for a periodic microstructure.

$$\overline{\overline{\chi}}_{e,0} = h \begin{pmatrix} \int_{\mathcal{D}} \varepsilon_r \boldsymbol{\mathcal{E}}_x \cdot \mathbf{x} \, \mathrm{d}\boldsymbol{\xi} - (\langle \varepsilon_r \rangle - 1) & \int_{\mathcal{D}} \varepsilon_r \boldsymbol{\mathcal{E}}_y \cdot \mathbf{x} \, \mathrm{d}\boldsymbol{\xi} & \int_{\mathcal{D}} \frac{1}{\varepsilon_0} \boldsymbol{\mathcal{D}}_z \cdot \mathbf{x} \, \mathrm{d}\boldsymbol{\xi} \\ \int_{\mathcal{D}} \varepsilon_r \boldsymbol{\mathcal{E}}_x \cdot \mathbf{y} \, \mathrm{d}\boldsymbol{\xi} & \int_{\mathcal{D}} \varepsilon_r \boldsymbol{\mathcal{E}}_y \cdot \mathbf{y} \, \mathrm{d}\boldsymbol{\xi} - (\langle \varepsilon_r \rangle - 1) & \int_{\mathcal{D}} \frac{1}{\varepsilon_0} \boldsymbol{\mathcal{D}}_z \cdot \mathbf{y} \, \mathrm{d}\boldsymbol{\xi} \\ \int_{\mathcal{D}} \varepsilon_r \boldsymbol{\mathcal{E}}_x \cdot \mathbf{z} \, \mathrm{d}\boldsymbol{\xi} & \int_{\mathcal{D}} \varepsilon_r \boldsymbol{\mathcal{E}}_y \cdot \mathbf{z} \, \mathrm{d}\boldsymbol{\xi} & \int_{\mathcal{D}} \frac{1}{\varepsilon_0} \boldsymbol{\mathcal{D}}_z \cdot \mathbf{z} \, \mathrm{d}\boldsymbol{\xi} \end{pmatrix}$$

Properties

We can prove some interesting properties on the susceptibilities :

- Real permittivity/permeabilities (no gain or absorption) lead to real susceptibilities.
- Symmetry of the tensors : $\overline{\overline{\chi}}_{e,0} = \overline{\overline{\chi}}_{e,0}^{\top}$ and $\overline{\overline{\chi}}_{m,0} = \overline{\overline{\chi}}_{m,0}^{\top}$.
- If the periodic microstructure is geometrically symmetric :

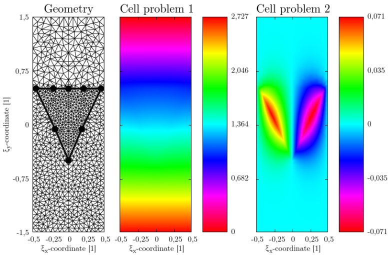
$$\overline{\overline{\chi}}_{e,0} = \begin{pmatrix} \chi^{xx}_{e,0} & 0 & 0 \\ 0 & \chi^{yy}_{e,0} & \chi^{yz}_{e,0} \\ 0 & \chi^{zy}_{e,0} & \chi^{zz}_{e,0} \end{pmatrix} \qquad \overline{\overline{\chi}}_{e,0} = \begin{pmatrix} \chi^{xx}_{e,0} & 0 & \chi^{xz}_{e,0} \\ 0 & \chi^{yy}_{e,0} & 0 \\ \chi^{zx}_{e,0} & 0 & \chi^{zz}_{e,0} \end{pmatrix} \qquad \overline{\overline{\chi}}_{e,0} = \begin{pmatrix} \chi^{xx}_{e,0} & \chi^{xy}_{e,0} & 0 \\ \chi^{yx}_{e,0} & \chi^{yy}_{e,0} & 0 \\ 0 & 0 & \chi^{zz}_{e,0} \end{pmatrix}$$

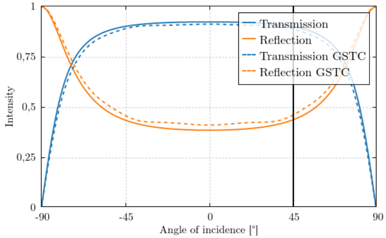
Along x (inplane). *Along y (inplane).* *Along z.*

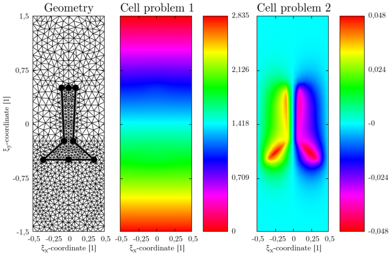
- If same symmetry in x and y then (uniaxial) :

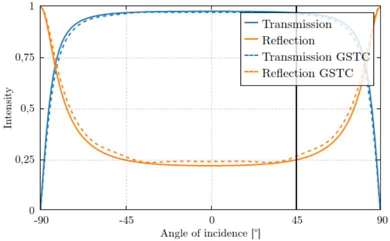
$$\overline{\overline{\chi}}_{e,0} = \begin{pmatrix} \chi^{1}_{e,0} & 0 & 0 \\ 0 & \chi^{1}_{e,0} & 0 \\ 0 & 0 & \chi^{2}_{e,0} \end{pmatrix}$$

Example in 2D









Simulation

In practice, GSTCs does not always lead to analytical solutions.

- More general sources.
- Other elements present in the simulation domain.
- Non-planar (curved) interfaces.
- Non periodic structures.

Variational formulation 1/2

The finite element method require the use of the variational formulation associated with Maxwell equations + GSTCs. We consider that $\mu_r = 1$.

$$\nabla \times \nabla \times \mathbf{E} - k^2 n^2 \mathbf{E} = 0$$

Variational formulation 1/2

The finite element method require the use of the variational formulation associated with Maxwell equations + GSTCs. We consider that $\mu_r = 1$.

$$\nabla \times \nabla \times \mathbf{E} \cdot \phi - k^2 n^2 \mathbf{E} \cdot \phi = 0$$

Variational formulation 1/2

The finite element method require the use of the variational formulation associated with Maxwell equations + GSTCs. We consider that $\mu_r = 1$.

$$\int_{\mathcal{D}} \nabla \times \nabla \times \mathbf{E} \cdot \phi - k^2 n^2 \mathbf{E} \cdot \phi \, \mathrm{d}\boldsymbol{x} = 0$$

Variational formulation 1/2

The finite element method require the use of the variational formulation associated with Maxwell equations + GSTCs. We consider that $\mu_r = 1$.

$$\int_{\mathcal{D}} \nabla \times \mathbf{E} \cdot \nabla \times \phi - k^2 n^2 \mathbf{E} \cdot \phi \, \mathrm{d}\boldsymbol{x} + \int_{\partial\mathcal{D}} \mathbf{n} \times \nabla \times \mathbf{E} \cdot \phi \, \mathrm{d}\boldsymbol{s} = 0$$

Variational formulation 1/2

The finite element method require the use of the variational formulation associated with Maxwell equations + GSTCs. We consider that $\mu_r = 1$.

$$\int_{\mathcal{D}} \nabla \times \mathbf{E} \cdot \nabla \times \phi - k^2 n^2 \mathbf{E} \cdot \phi \, \mathrm{d}\boldsymbol{x} - i\omega\mu_0 \int_{\partial\mathcal{D}} \mathbf{n} \times \mathbf{H} \cdot \phi \, \mathrm{d}\boldsymbol{s} = 0$$

Variational formulation 1/2

The finite element method require the use of the variational formulation associated with Maxwell equations + GSTCs. We consider that $\mu_r = 1$.

$$\int_{\mathcal{D}} \nabla \times \mathbf{E} \cdot \nabla \times \phi - k^2 n^2 \mathbf{E} \cdot \phi \, \mathrm{d}\boldsymbol{x} - i\omega\mu_0 \int_{\partial\mathcal{D}} \mathbf{n} \times \mathbf{H} \cdot \phi \, \mathrm{d}\boldsymbol{s} = 0$$

Thus, on the interface S^- (S^+, S^- are each sides of the GSTCs) we need :

$$-i\omega\mu_0 \int_{S^-} [\![\mathbf{n} \times \mathbf{H} \cdot \phi]\!] \, \mathrm{d}\boldsymbol{s} = -i\omega\mu_0 \int_{S^-} \mathbf{n} \times [\![\mathbf{H}]\!] \cdot \{\phi\} + \mathbf{n} \times \{\mathbf{H}\} \cdot [\![\phi]\!] \, \mathrm{d}\boldsymbol{s}.$$

If we consider the inversion method or thin layers :

$$\mathbf{n} \times [\![\mathbf{E}]\!] = -i\omega\mu_0 \left(\overline{\overline{\chi}}_{m,0} \{\mathbf{H}\}\right)_{\parallel} \quad \Rightarrow \quad \mathbf{n} \times \{\mathbf{H}\} = \mathbf{n} \times \frac{-1}{i\omega\mu_0} \overline{\overline{\chi}}_{m,0}^{-1} \mathbf{n} \times [\![\mathbf{E}]\!],$$

$$\mathbf{n} \times [\![\mathbf{H}]\!] = +i\omega\varepsilon_0 \left(\overline{\overline{\chi}}_{e,0} \{\mathbf{E}\}\right)_{\parallel}.$$

Variational formulation 2/2

For general GSTCs obtaining $\mathbf{n} \times \{\mathbf{H}\}$ requires an additional step.

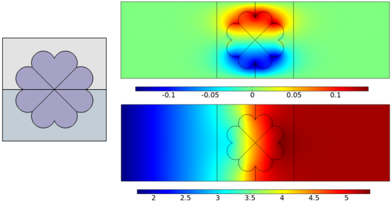
$$\mathbf{n} \times [\![\mathbf{E}]\!] = \nabla_{\|} \times \left(\overline{\overline{\chi}}_{e,0} \{\mathbf{E}\}\right)_{\perp} - i\omega\mu_0 \left(\overline{\overline{\chi}}_{m,0} \{\mathbf{H}\}\right)_{\|},$$
$$\mathbf{n} \times [\![\mathbf{H}]\!] = \nabla_{\|} \times \left(\overline{\overline{\chi}}_{m,0} \{\mathbf{H}\}\right)_{\perp} + i\omega\varepsilon_0 \left(\overline{\overline{\chi}}_{e,0} \{\mathbf{E}\}\right)_{\|},$$

Using that $\mathbf{E} = \nabla \times \mathbf{H}/(i\omega\varepsilon_0\varepsilon_r)$ and $\mathbf{H} = \nabla \times \mathbf{E}/(-i\omega\mu_0)$:

$$\nabla_{\|} \times \left(\frac{\overline{\overline{\chi}}_{e,0}}{\varepsilon_r} \nabla_{\|} \times \{\mathbf{H}_{\|}\}\right) + k^2 \left(\overline{\overline{\chi}}_{m,0} \{\mathbf{H}_{\|}\}\right) = i\omega\varepsilon_0 [\![\mathbf{E}]\!] \times \mathbf{n},$$
$$\nabla_{\|} \times \left(\overline{\overline{\chi}}_{m,0} \nabla_{\|} \times \{\mathbf{E}_{\|}\}\right) + k^2 \left(\overline{\overline{\chi}}_{e,0} \{\mathbf{E}_{\|}\}\right) = \omega\mu_0 [\![\mathbf{H}]\!] \times \mathbf{n},$$

The GSTCs may be seen as surfacic Maxwell equations for the mean of the tangential components $\left\{\mathbf{E}_{\|}\right\}, \left\{\mathbf{H}_{\|}\right\}$ with electric and magnetic sources.

2D example : homogenized clovers



$$\overline{\overline{\chi}}_{e,0} = \begin{pmatrix} -0.046 & 0 & 0 \\ 0 & 0.312 & 0 \\ 0 & 0 & 0.604 \end{pmatrix}$$

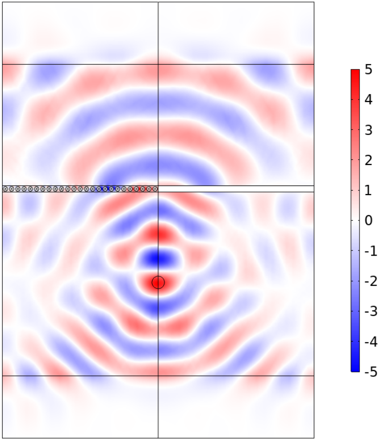
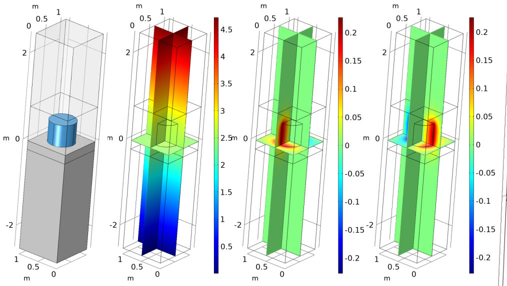


Fig. Simulation of a 2D metasurface with "clover-like" elements in silicon, substrate of silica, surrounded by air (thickness $h = \lambda/10$, period $d_x = \lambda/10$).

3D example : homogenized cylinders



$$\overline{\overline{\chi}}_{e,0} = \begin{pmatrix} -0.072 & 0 & 0 \\ 0 & 0.142 & 0 \\ 0 & 0 & 0.142 \end{pmatrix}$$

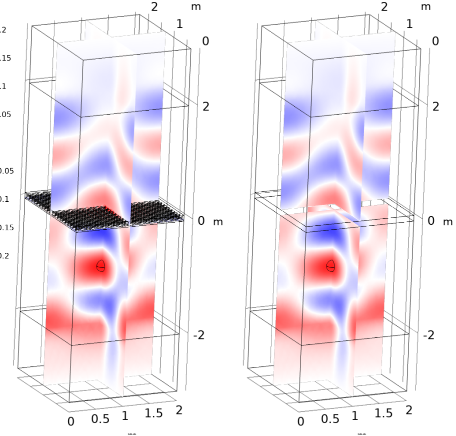
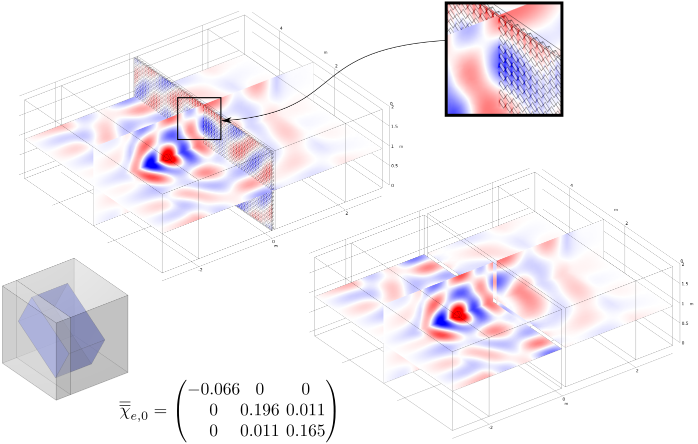


Fig. Simulation of a 3D metasurface with cylinders in silicon and fully surrounded by air (thickness $h = \lambda/10$, period $d_x = \lambda/10$).

3D examples : homogenized cuboids



$$\overline{\overline{\chi}}_{e,0} = \begin{pmatrix} -0.066 & 0 & 0 \\ 0 & 0.196 & 0.011 \\ 0 & 0.011 & 0.165 \end{pmatrix}$$

Fig. Simulation of a 3D metasurface with rotated cuboids in silicon and fully surrounded by air (thickness $h = \lambda/10$, period $d_x = \lambda/10$).

Time-dependent examples

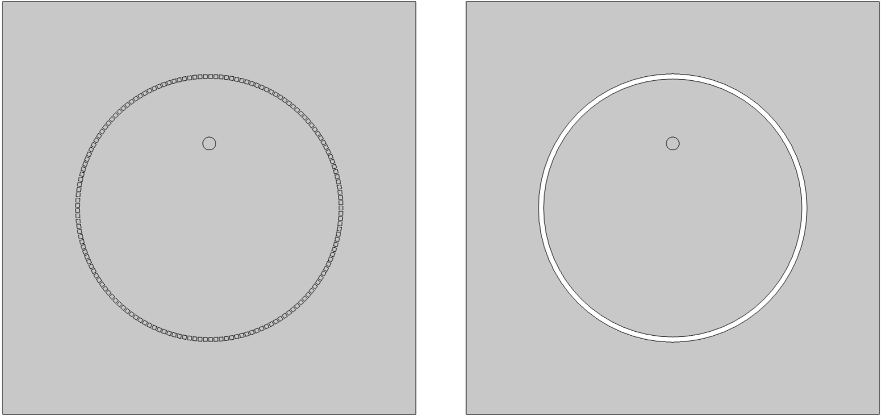


Fig. Time-domain conformal GSTC with Discontinuous Galerkin method.

Time-dependent examples

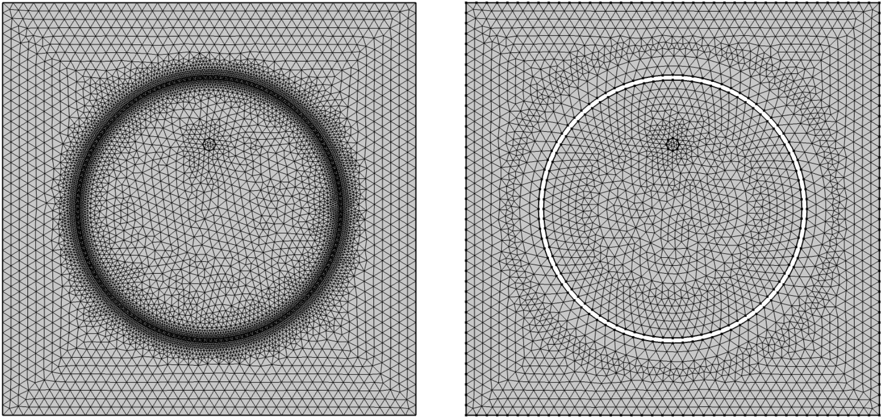


Fig. Time-domain conformal GSTC with Discontinuous Galerkin method.

3D time-dependent example

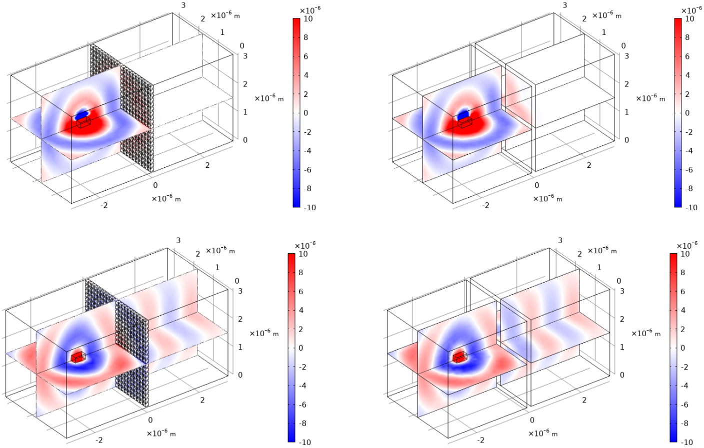
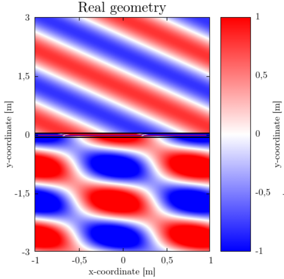
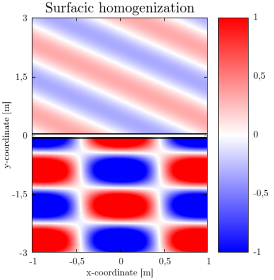
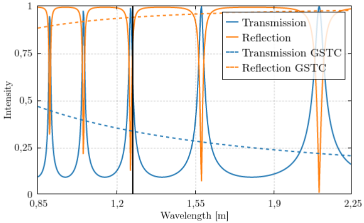


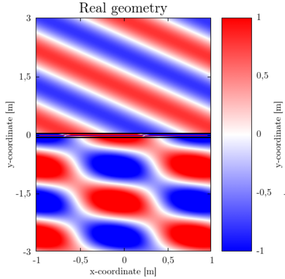
Fig. Time-domain three dimensional GSTC with Discontinuous Galerkin method.

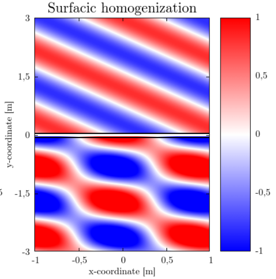
Resonant inclusions

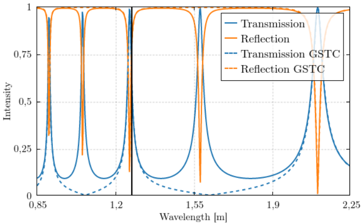












Quasi-periodic & optimization

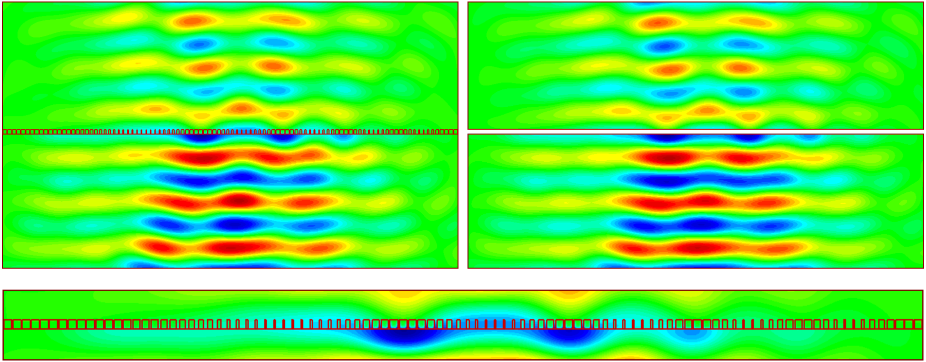


Fig. Quasi-periodic homogenization gives the (non constant) effective properties associated with any non-periodic metasurface.

Conclusion

- GSTCs provides the transition conditions verified by the fields on an infinitely thin metasurface using surfacic material properties.
- The homogenization theory gives tools to find such surfacic quantities associated with a real (deeply subwavelength) metasurface.
- The finite element method can take into account the boundary conditions arising from GSTCs, thus reducing the cost of simulating thin metasurfaces.

$\rightarrow$ All the methods shown in this presentation can (easily !) be adapted to any other linear partial differential equations.

Thanks for your attention !