Wave propagation in disordered and nonlinear phononic chains

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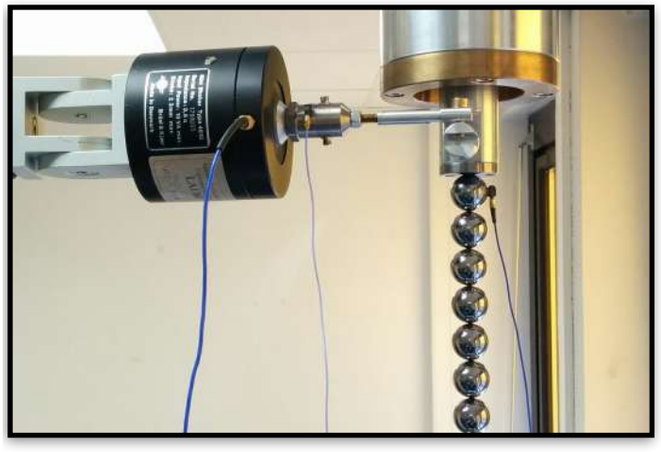
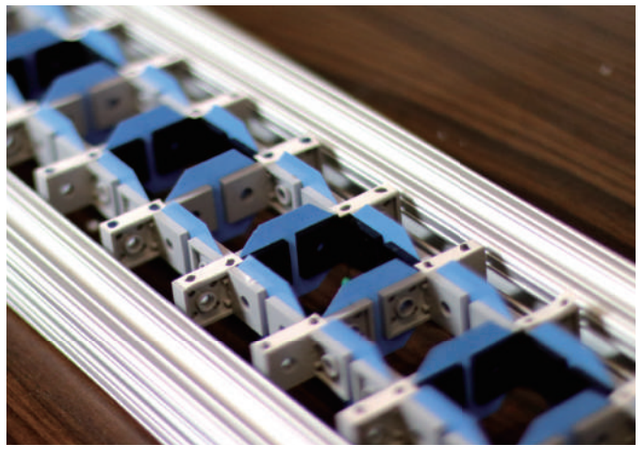
Mechanical lattices

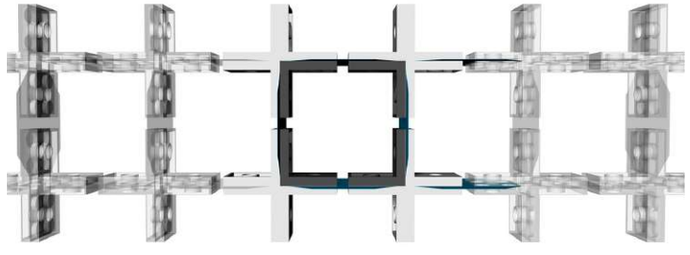
Granular crystals



*F. Allein, PhD Thesis (2017)

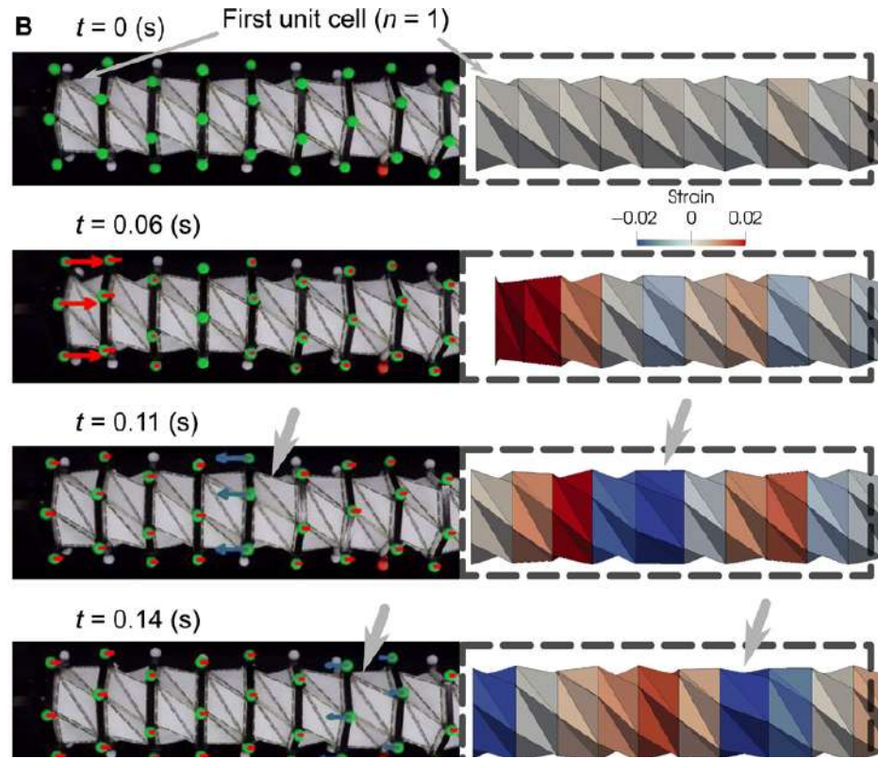
Flexible elastic metamaterials





B. Deng, V. Tournat, P. Wang and K. Bertoldi
Phys. Rev. Lett. **122,** (2019)

Origami lattices



H. Yasuda *et.al,* Sci. Adv. **5** (2019)

Tunable dispersion relation characteristics

Tunable nonlinearity

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Outline

- Motivation and Introduction

- A disordered granular lattice vs the FPU model
  - Delocalisation due to nonlinearity
  - Chaos and equipartition

- 1D disorder lattices with 2-DOF (the effect of rotations)
  - Slower spreading due to rotations
  - Polarised spreading

- Nonlinear flexible metamaterials with strong disorder
  - Root to almost ballistic spreading

- Perspectives

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Outline

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- Perspectives

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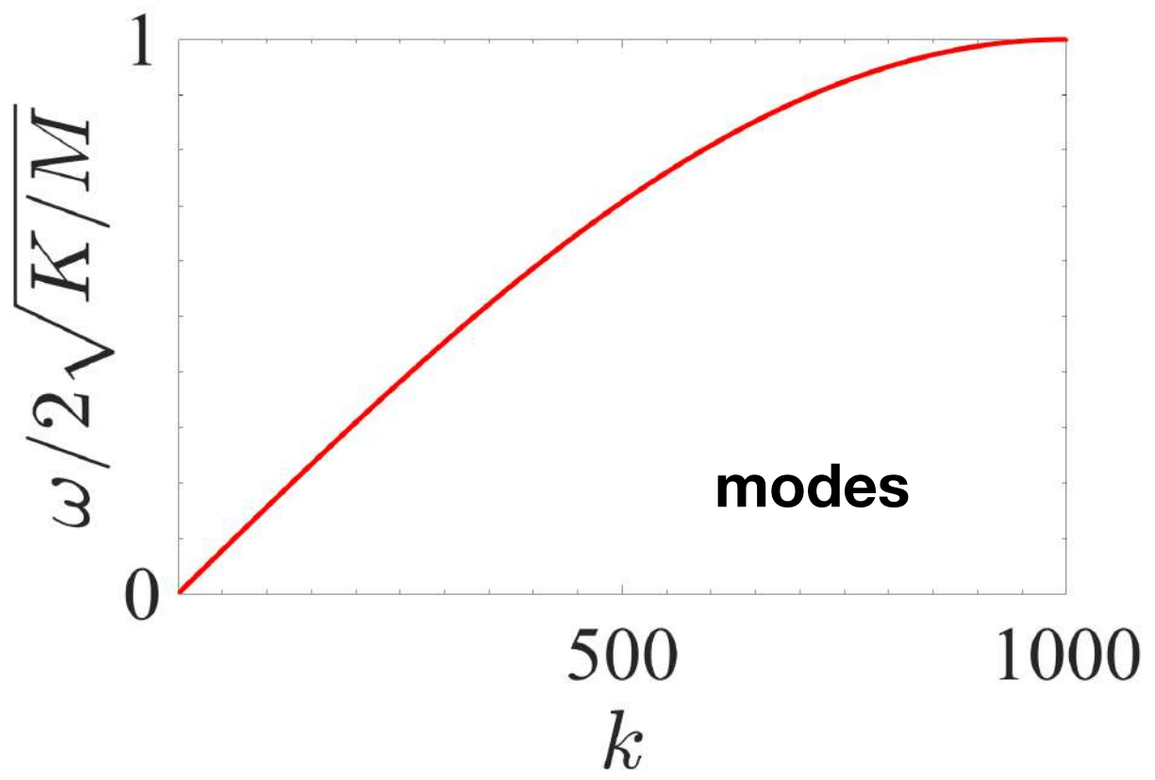
1D linear periodic mass spring lattice

Hamiltonian

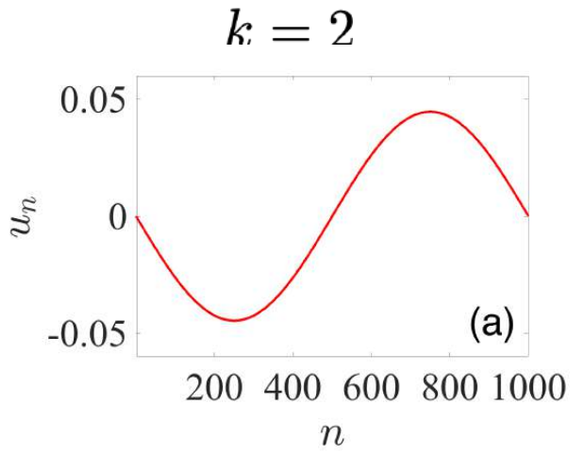
$$H = \sum_{n=1}^{N} \frac{p_n^2}{2m_n} + \frac{K}{2}(u_{n+1} - u_n)^2$$

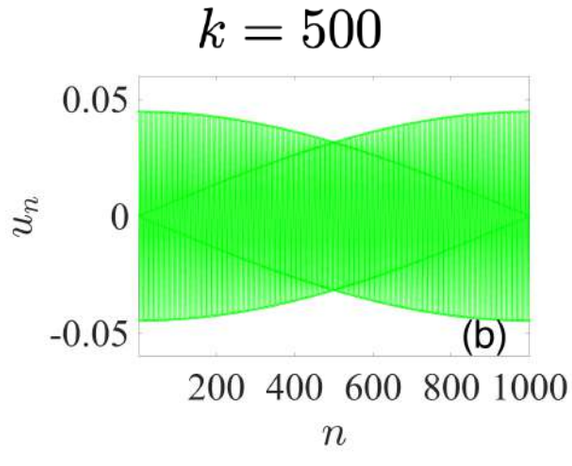
Eigenvalue problem

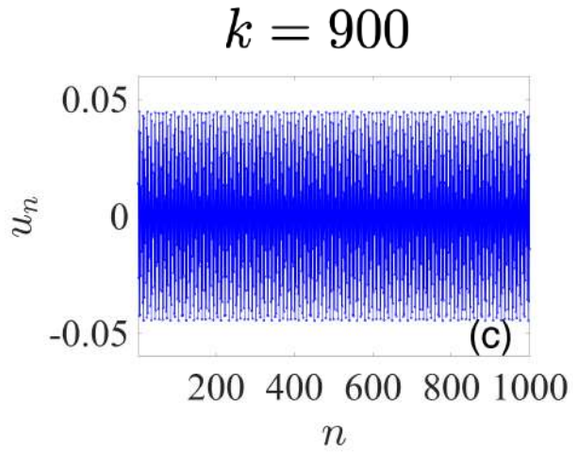
$$-\omega_k^2 \mathbf{M}\mathbf{U}_k = \mathbf{K}\ \mathbf{U}_k$$

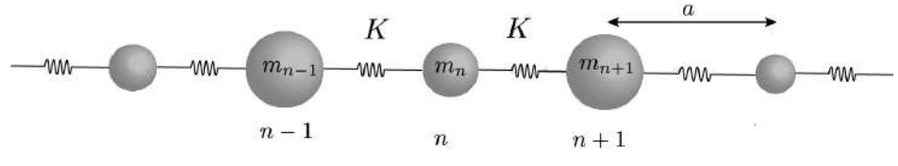


All eigenmodes are extended









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1D linear *disordered* mass spring lattice

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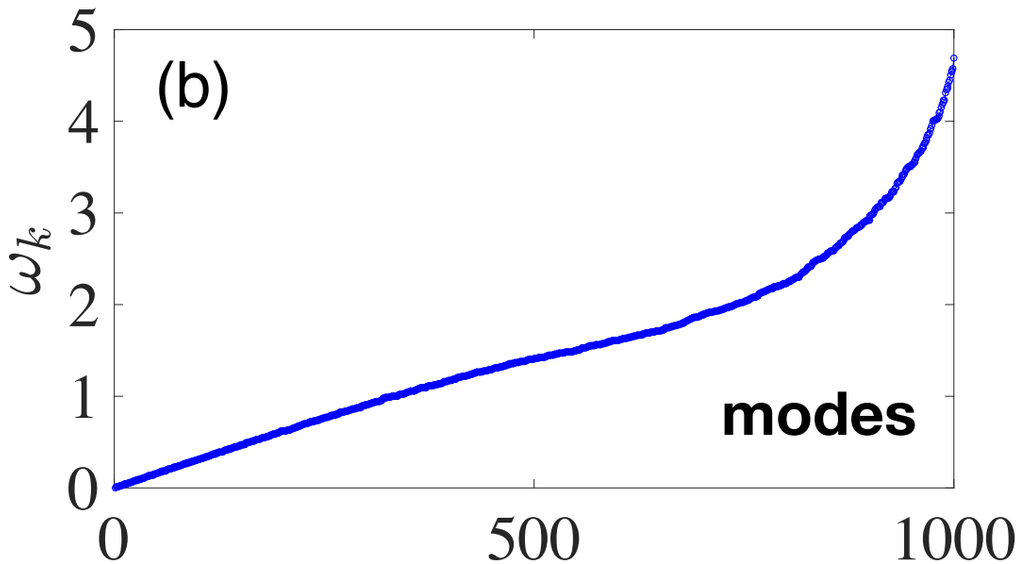

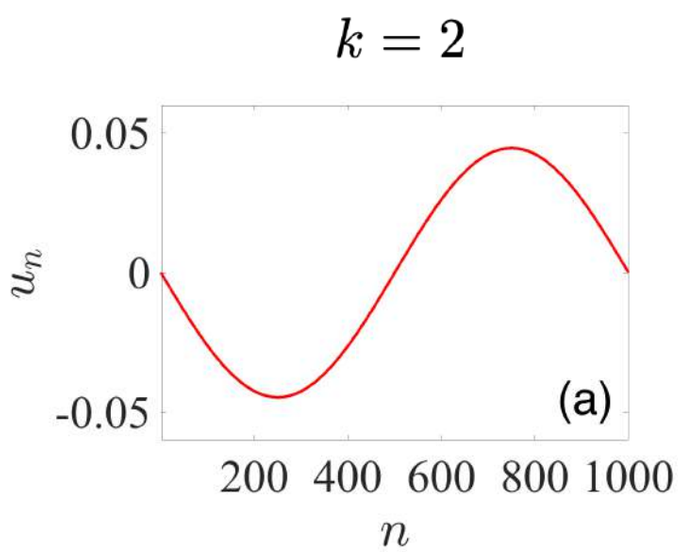

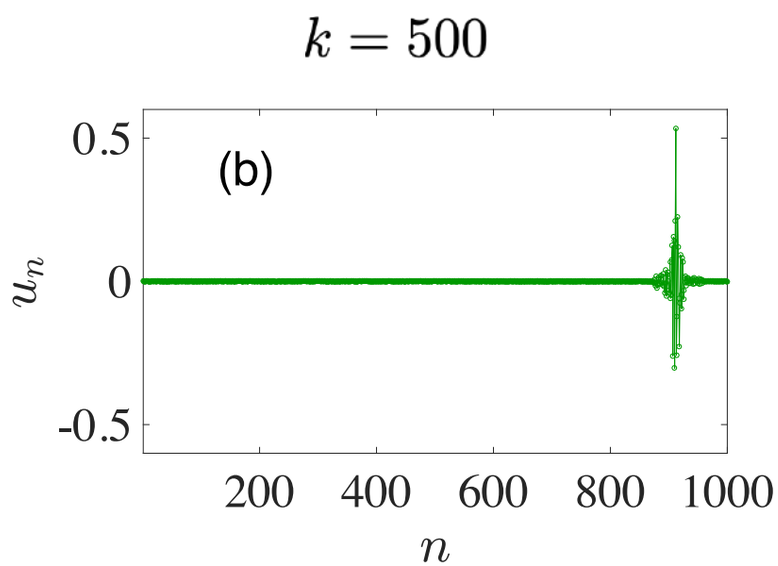

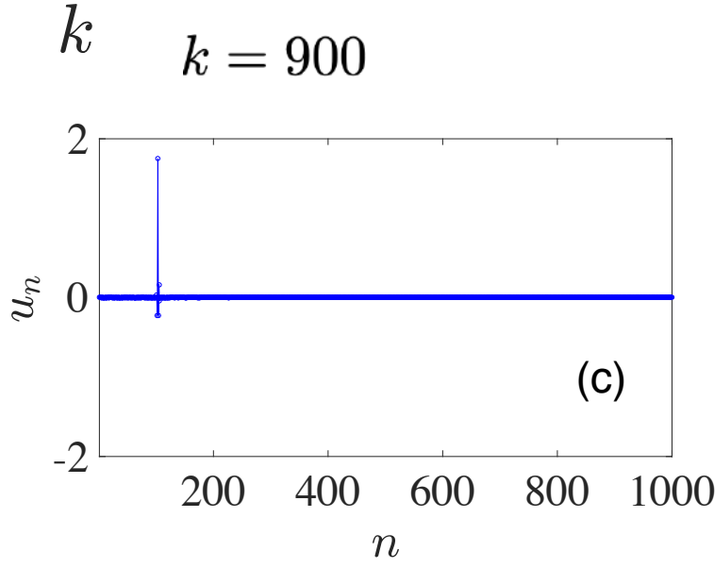

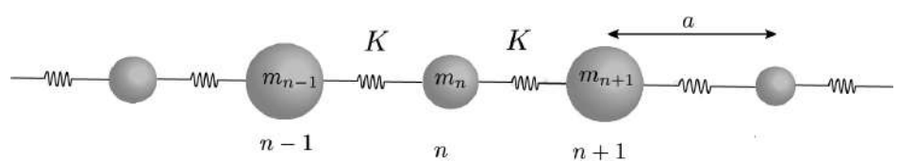

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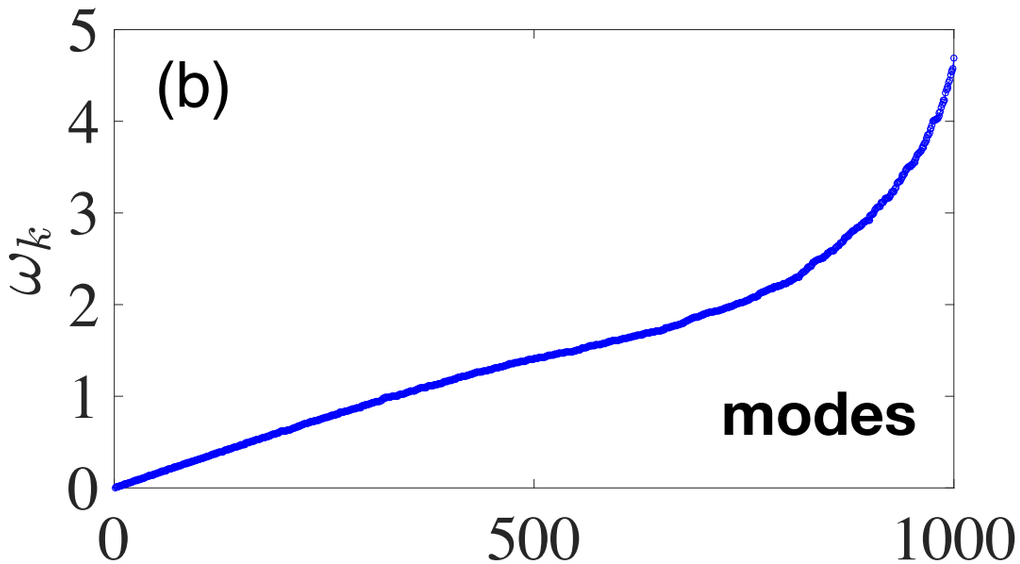
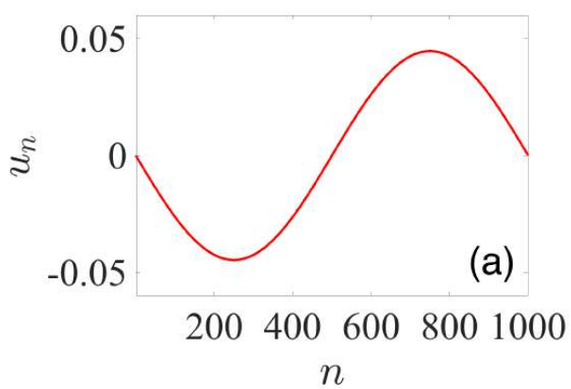
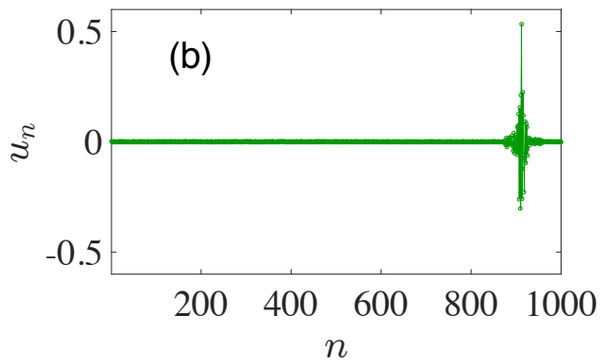


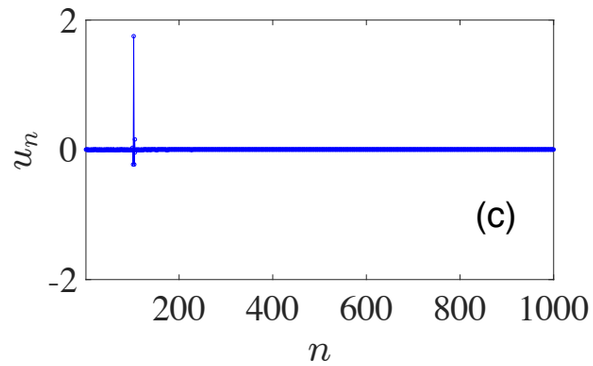
$k = 2$



$k = 500$

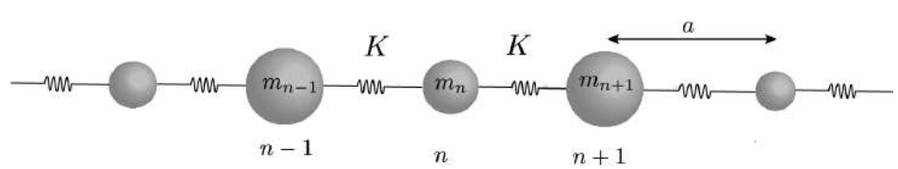


k $k = 900$



For $n \to \infty$ there are $\sqrt{N}$ extended modes

* H. Matsuda and K. Ishii, Prog. Theor. Phys. Suppl. 45, 56 (1970)
* K. Ishii, Prog. Theor. Phys. Suppl. 53, 77 (1973)



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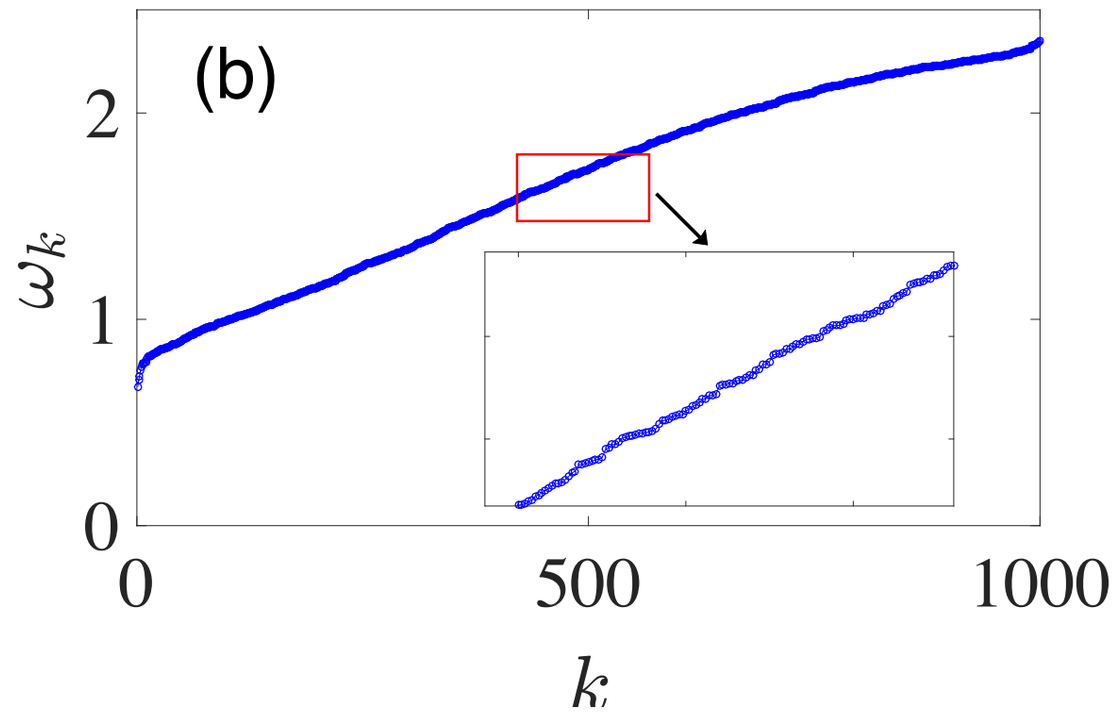
1D linear *disordered* Klein-Gordon

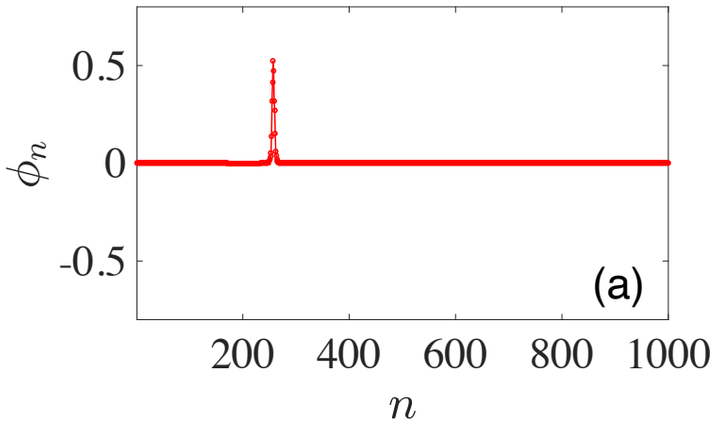
Hamiltonian

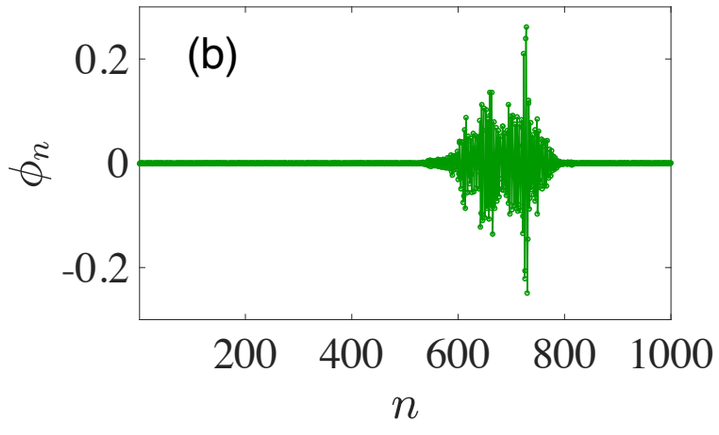
$$H = \sum_n \frac{1}{2}\pi_n^2 + \frac{1}{2}(\phi_{n+1} - \phi_n)^2 + \frac{1}{2}m_n\phi_n^2$$

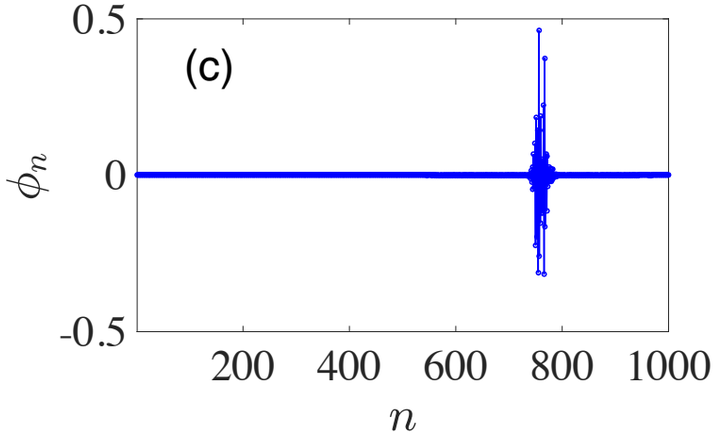
Eigenvalue problem

$$-\omega_k^2 \mathbf{\Phi}_k = \mathbf{K}(m_n)\mathbf{\Phi}_k$$

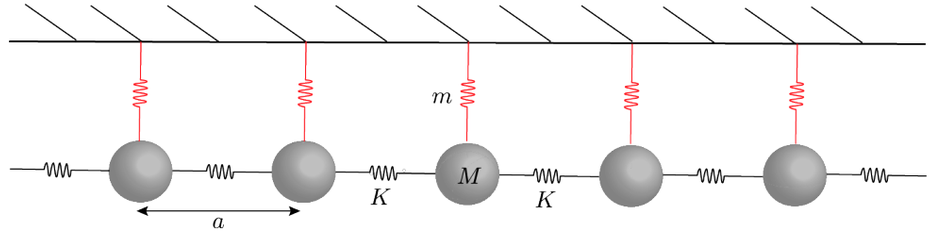








All modes are localised



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1D linear *disordered* Klein-Gordon/Anderson models

Anderson model

$$i\dot{\psi}_l = \epsilon_l \psi_l - \psi_{l+1} - \psi_{l-1}$$

Discrete K-G

$$\ddot{u}_l = -\bar{\epsilon}_l u_l - u^3 + \frac{1}{W}(u_{l+1} + u_{l-1} - 2u_l)$$

	Anderson	KG
On-site energies	$\epsilon_l \in [-\frac{W}{2}, \frac{W}{2}]$	$\bar{\epsilon}_l \in [\frac{1}{2}, \frac{3}{2}]$
Spectrum	$\lambda_\nu \in [-2 - \frac{W}{2}, 2 + \frac{W}{2}]$	$\omega_\nu^2 \in [\frac{1}{2}, \frac{3}{2} + \frac{4}{W}]$
Spectrum width Δ	$\Delta_D = W + 4$	$\Delta_K = \frac{W+4}{W}$
Localization volume V	$\begin{cases} W \to 0 & V = \frac{330}{W^2} \\ W \to \infty & V \sim 1 \end{cases}$	

All modes are localised

- Ch. Skokos, D.O. Krimer, S. Komeneas, and S. Flach, Phys. Rev. E **79**, 056211 (2009)

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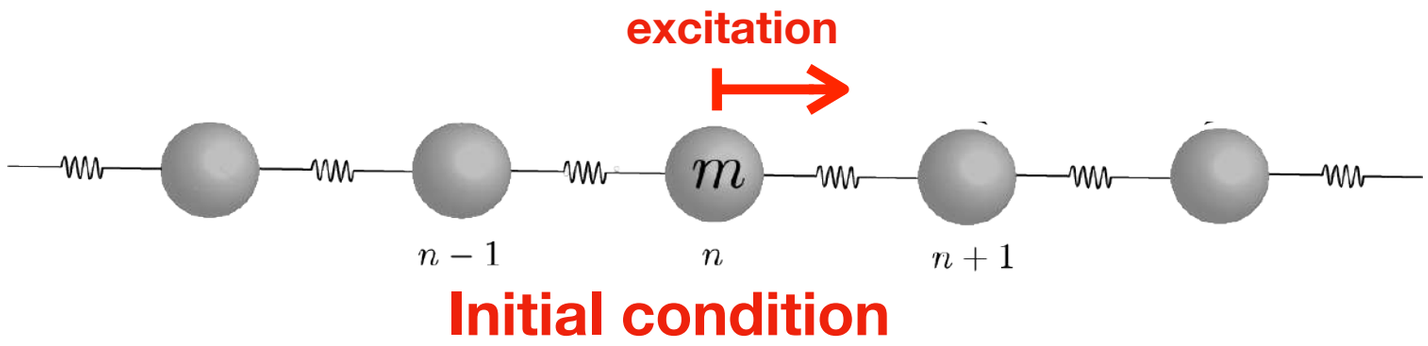
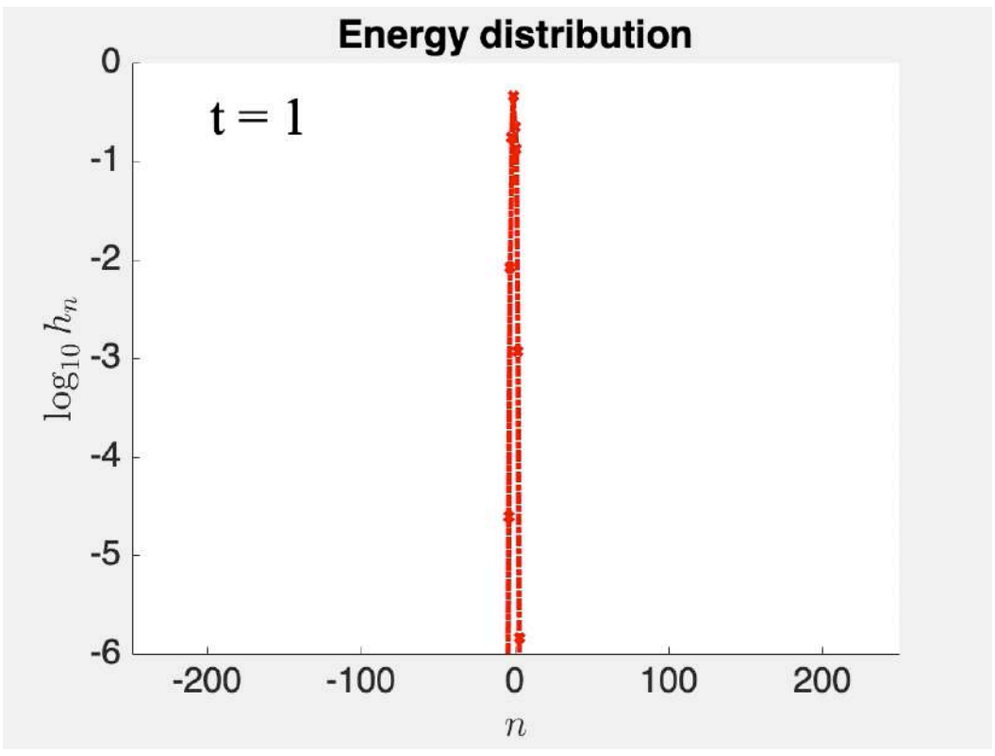
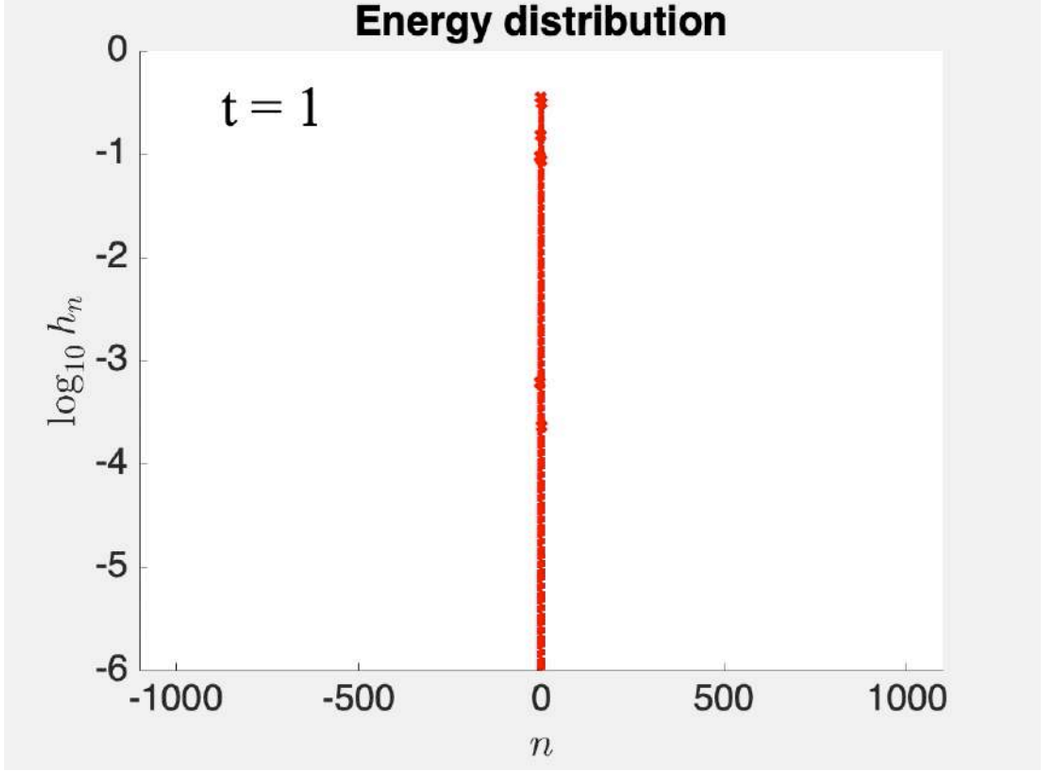
Single site excitations

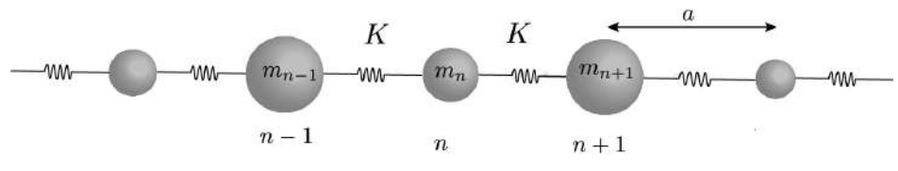


Homogeneous



Disordered





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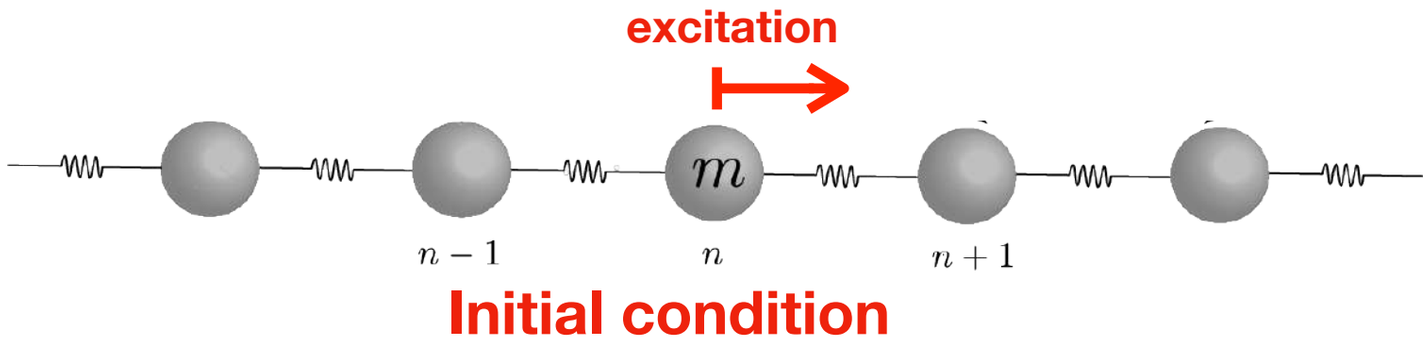
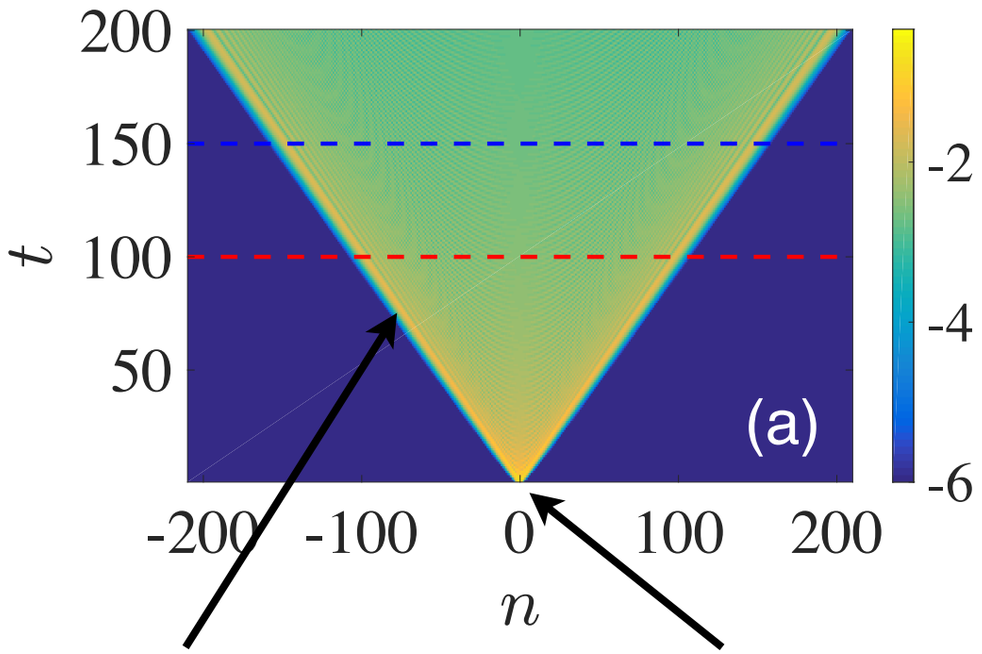
Single site excitations



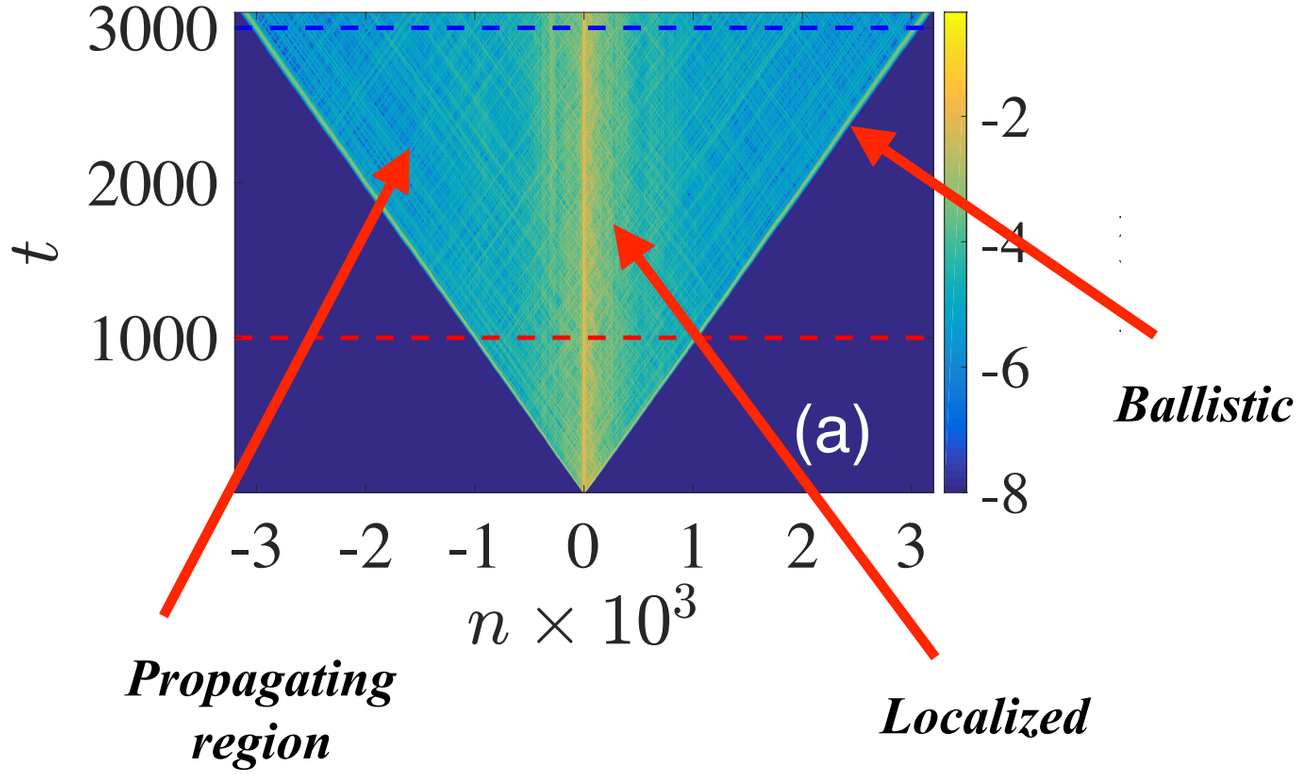
Homogeneous



Propagating front

Initial excitation site

Disordered



Propagating region

Localized

Ballistic

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Tools to study the spreading

Participation number: $$P = \frac{1}{\sum_{n=1}^{N} h_n^2} \qquad h_n = H_n/H$$

Second moment : $$m_2 = \sum_{1}^{N} |n - n_0|^2 h_n \qquad m_2(t) \propto t^{\beta} \qquad \beta = \frac{d \log_{10} m_2(t)}{d \log_{10} t}$$

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Tools to study the spreading

Participation number: $P = \dfrac{1}{\sum_{n=1}^{N} h_n^2}$ $\quad h_n = H_n/H$

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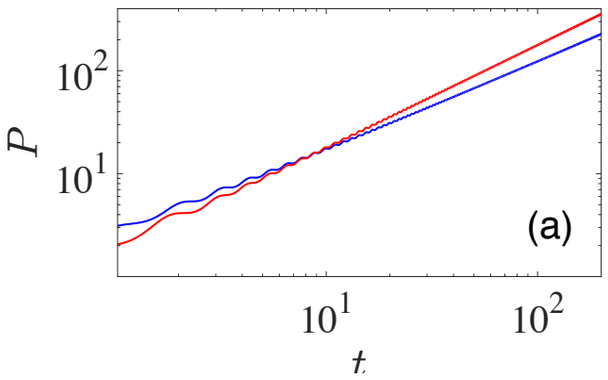
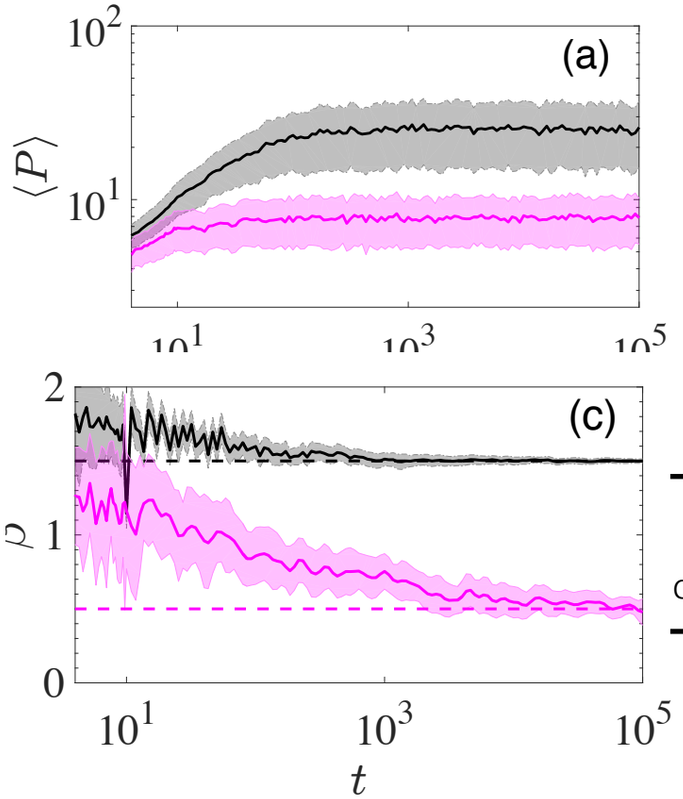
Homogeneous



Disordered



P.K. Datta & K. Kundu, PRB (1995)

S. Lepri et al, PRE (2010)

momentum → $\beta = 1.5$ superdiffusive

displacement → $\beta = 0.5$ subdiffusive

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What is the role of nonlinearity in the dynamics

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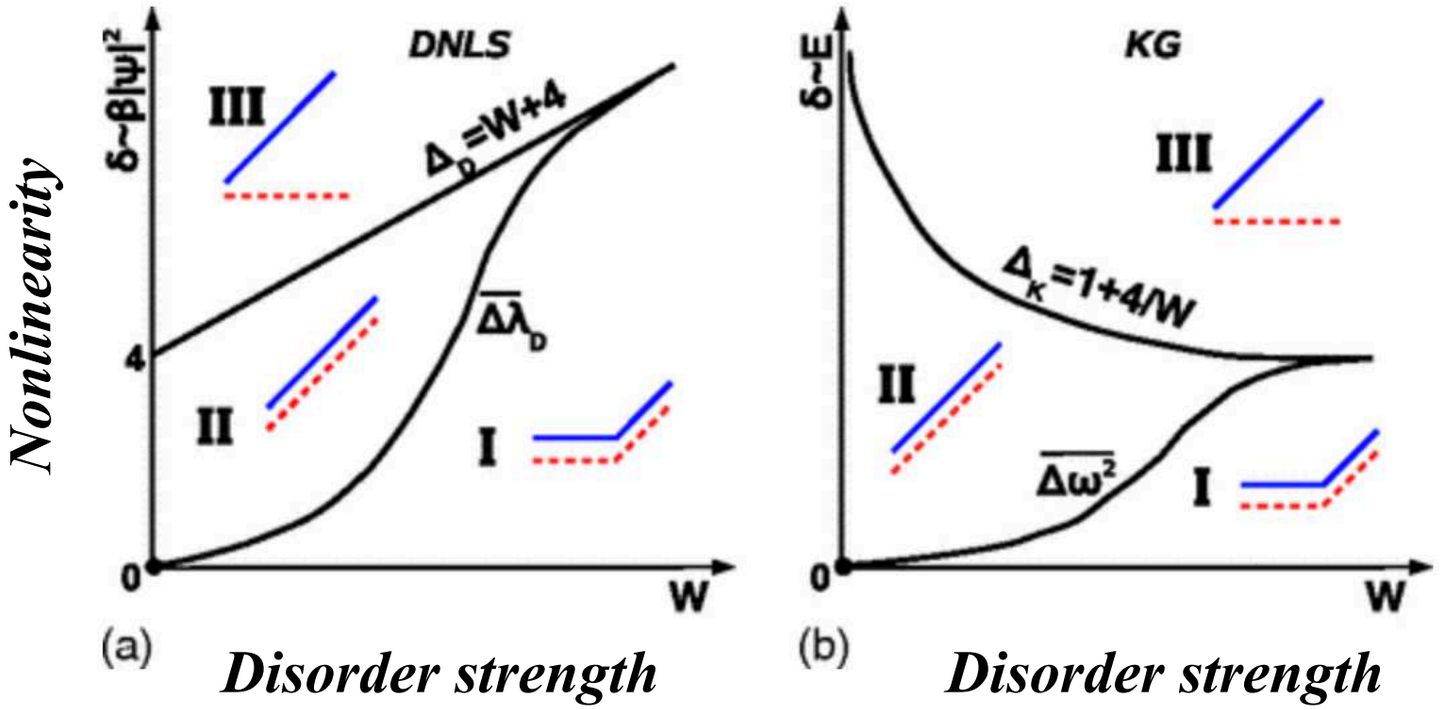
Theory on nonlinear and disorder lattices in 1D

DNLS

$$i\dot{\psi}_l = \epsilon_l \psi_l + \beta|\psi_l|^2\psi_l - \psi_{l+1} - \psi_{l-1}$$

DKG

$$\ddot{u}_l = -\tilde{\epsilon}_l u_l - u_l^3 + \frac{1}{W}(u_{l+1} + u_{l-1} - 2u_l)$$



II: Subdiffusive spreading + localization

II: Subdiffusive spreading

I :Localization

- Ch. Skokos, D.O. Krimer, S. Komeneas, and S. Flach, Phys. Rev. E **79**, 056211 (2009)

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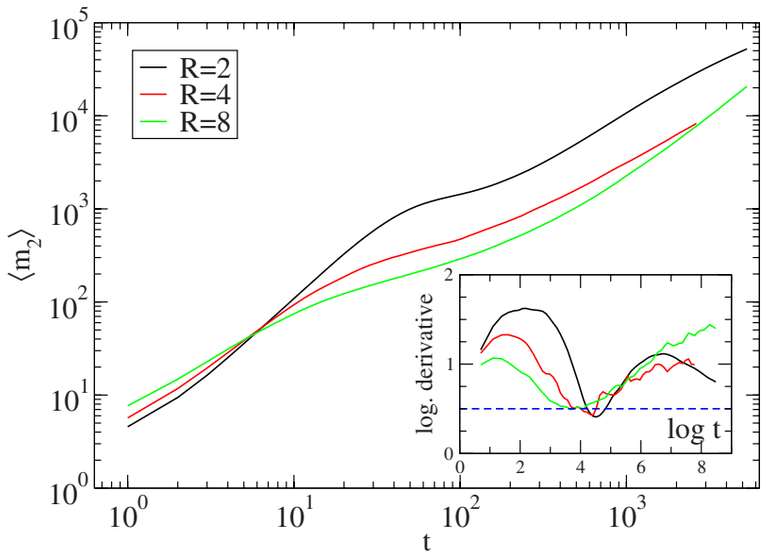
Other nonlinear models closer to mechanical lattices

FPU-β

$$m\ddot{u}_n = K_n(u_{n+1} - u_n) - K_{n-1}(u_n - u_{n-1}) + G(u_{n+1} - u_n)^3 - G(u_{n-1} - u_n)^3.$$

Inconclusive



S. Lepri, R. Schilling, and S. Aubry
Phys. Rev. E **82**, 056602 (2010)

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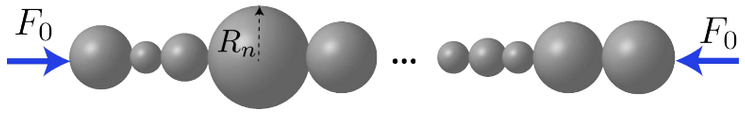
Other nonlinear models closer to mechanical lattices

FPU-β

$$m\ddot{u}_n = K_n(u_{n+1}-u_n) - K_{n-1}(u_n-u_{n-1}) + G(u_{n+1}-u_n)^3 - G(u_{n-1}-u_n)^3.$$

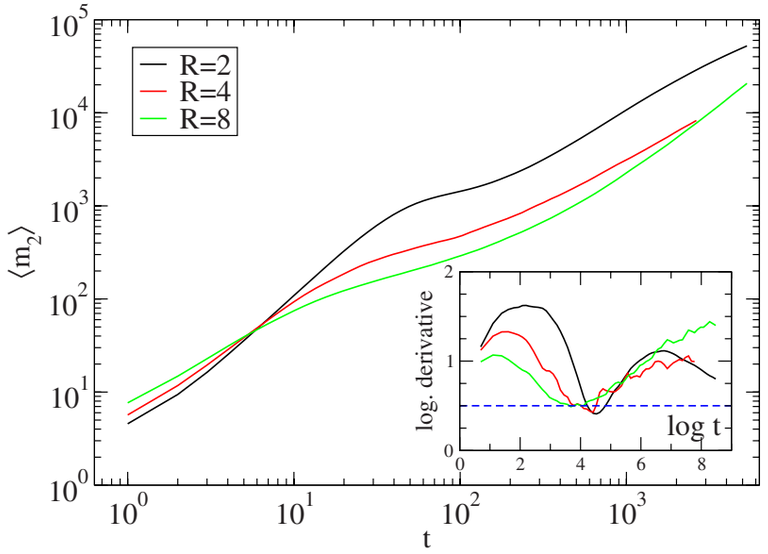
Hertzian contacts-granular lattice



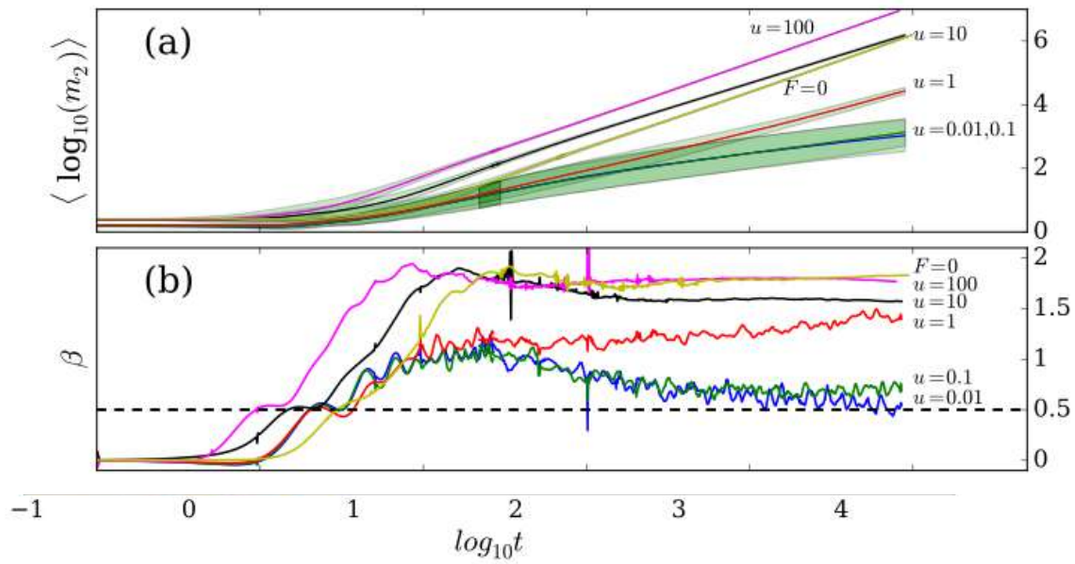
$$m_n\ddot{u}_n = A_n[\delta_n + u_{n-1} - u_n]_+^{\frac{3}{2}} - A_{n+1}[\delta_{n+1} + u_n - u_{n+1}]_+^{\frac{3}{2}}$$

$$[x]_+ = 0 \quad \text{if} \quad x < 0$$

Inconclusive



S. Lepri, R. Schilling, and S. Aubry
Phys. Rev. E **82**, 056602 (2010)



Inconclusive in the weakly nonlinear regime
Ballistic spreading in the strongly nonlinear

VA, G. Theocharis, and Ch. Skokos
Phys. Rev. E **93**, 022903 (2016)

A. J. Martínez, P. G. Kevrekidis, and M. A. Porter
Phys. Rev. E **93**, 022902 (2016)

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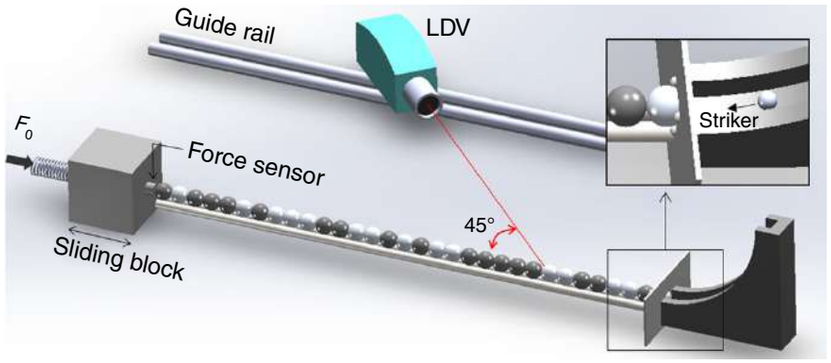
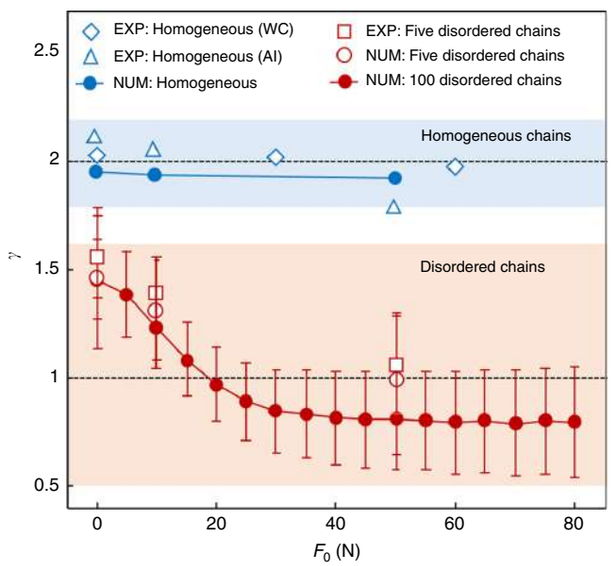
Other nonlinear models closer to mechanical lattices

ARTICLE
DOI: 10.1038/s41467-018-03015-3 OPEN

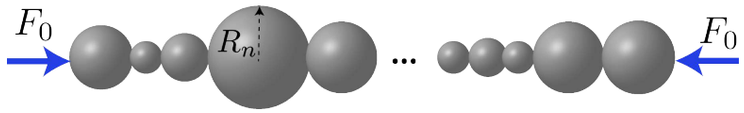
Direct measurement of superdiffusive energy transport in disordered granular chains

Eunho Kim[1,2], Alejandro J. Martínez[3], Sean E. Phenisee[1], P.G. Kevrekidis[4], Mason A. Porter[3,5,6] & Jinkyu Yang[1]



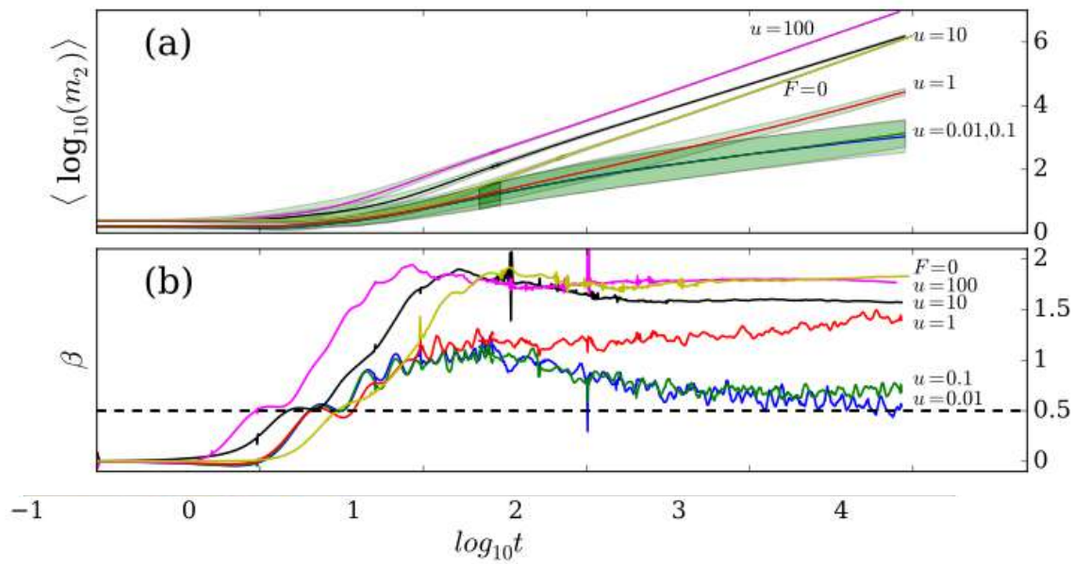


Hertzian contacts- granular lattice



$$m_n\ddot{u}_n = A_n[\delta_n + u_{n-1} - u_n]_+^{\frac{3}{2}} - A_{n+1}[\delta_{n+1} + u_n - u_{n+1}]_+^{\frac{3}{2}}$$

$$[x]_+ = 0 \quad \text{if} \quad x < 0$$



Inconclusive in the weakly nonlinear regime
Ballistic spreading in the strongly nonlinear

VA, G. Theocharis, and Ch. Skokos
Phys. Rev. E **93**, 022903 (2016)

A. J. Martínez, P. G. Kevrekidis, and M. A. Porter
Phys. Rev. E **93**, 022902 (2016)

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Outline

- Motivation and Introduction

- A finite disordered granular lattice vs the FPU model
  - Delocalisation due to nonlinearity
  - Chaos and equipartition

- 1D disorder lattices with 2-DOF (the effect of rotations)
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- Perspectives

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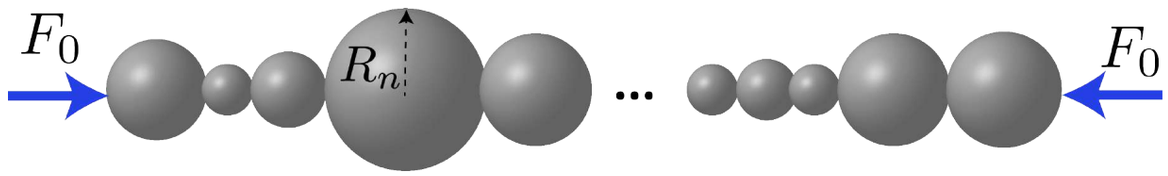
Model for granular crystal lattice



$$H = \sum_{n=1}^{N} H_n = \sum_{n=1}^{N} \frac{p_n^2}{2m_n} + V_n^{(Hz,F)} \qquad p_n = m_n \dot{u}_n$$

$$m_n \ddot{u}_n = A_n [\delta_n + u_{n-1} - u_n]_+^{\frac{3}{2}} - A_{n+1} [\delta_{n+1} + u_n - u_{n+1}]_+^{\frac{3}{2}}$$

Connection with the FPUT model

$u_n/\delta_{n,n+1} \ll 1$ ↓ small amplitude limit

$$m_n \ddot{u}_n = \sum_{k=2}^{4} [K_{n+1}^{(k)} (u_{n+1} - u_n)^{k-1} - K_n^{(k)} (u_n - u_{n-1})^{k-1}]$$

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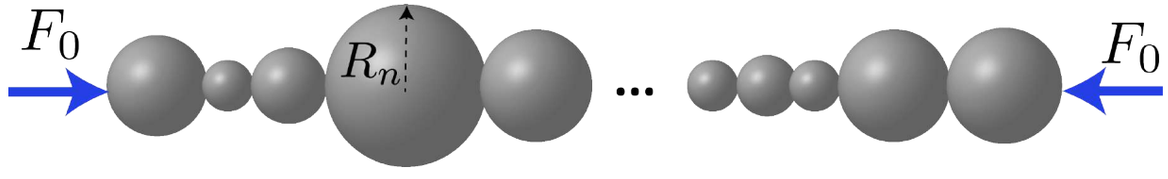
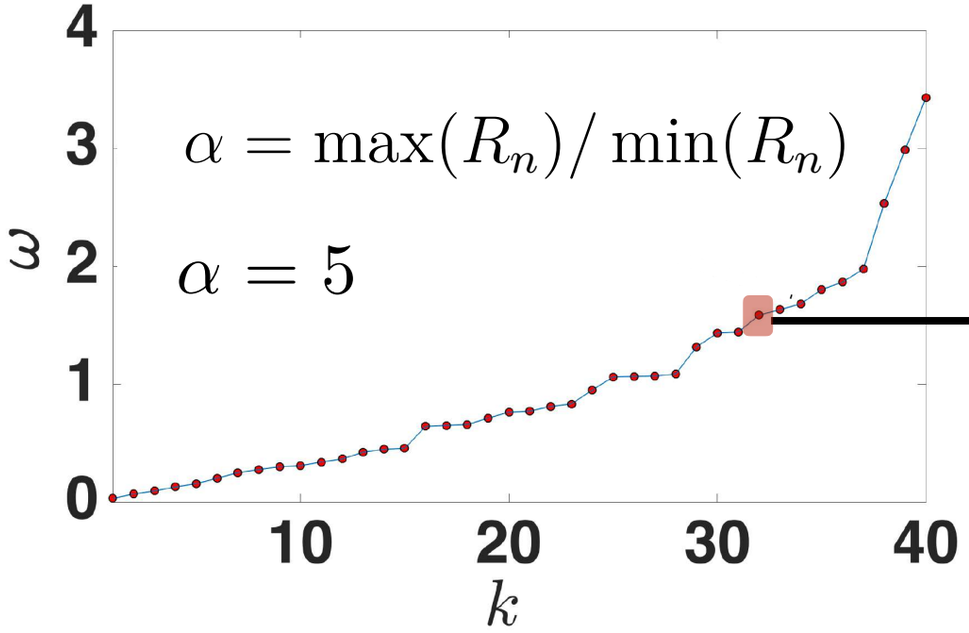
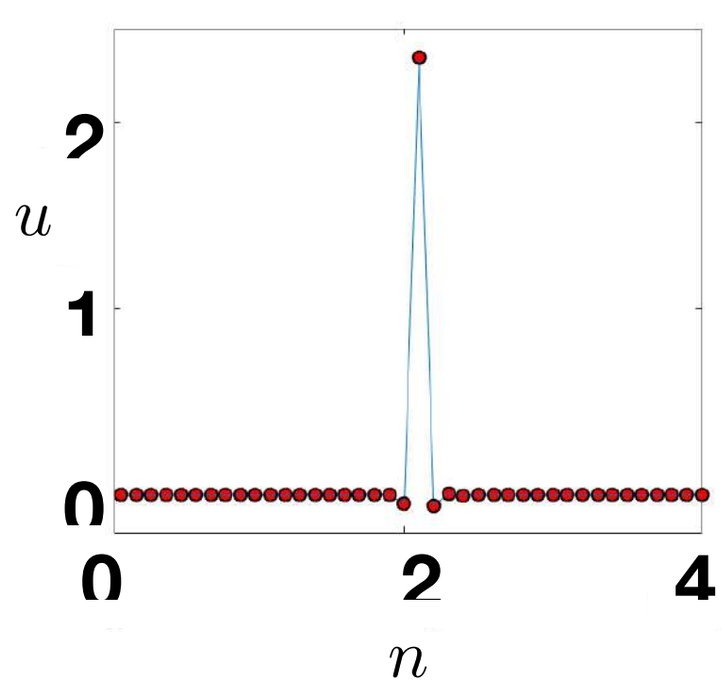
Effect of nonlinerity and comparison with FPUT model



1) Study the fate of a highly localized mode in a finite lattice

2) Compare with the more general model of FPUT





We excite the central particle with increasing energy (nonlinearity)

Increasing energy

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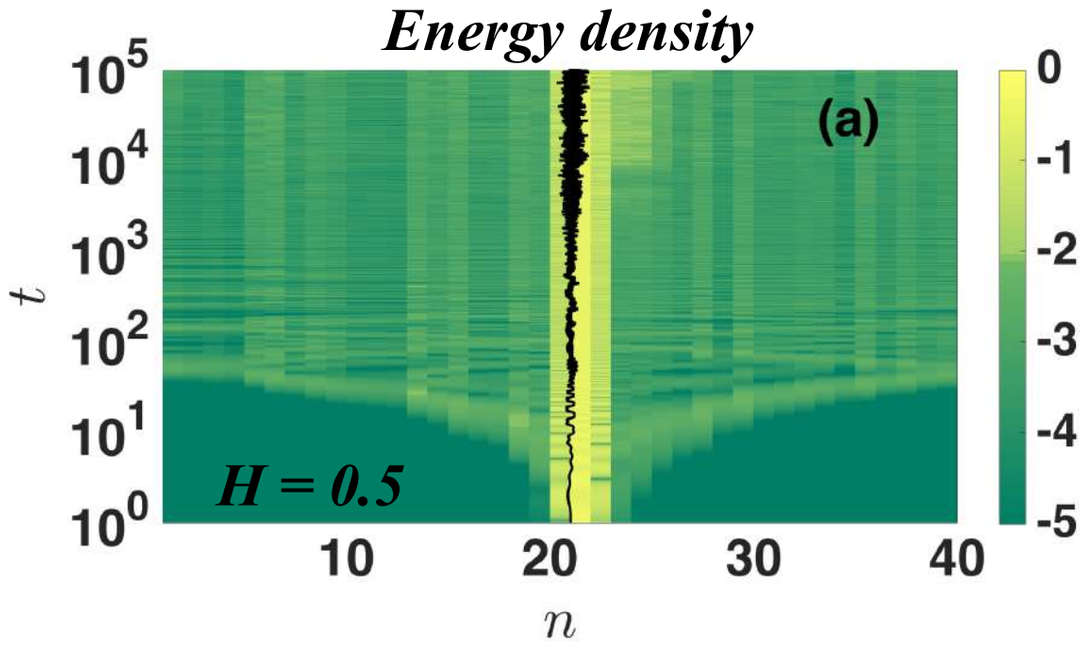
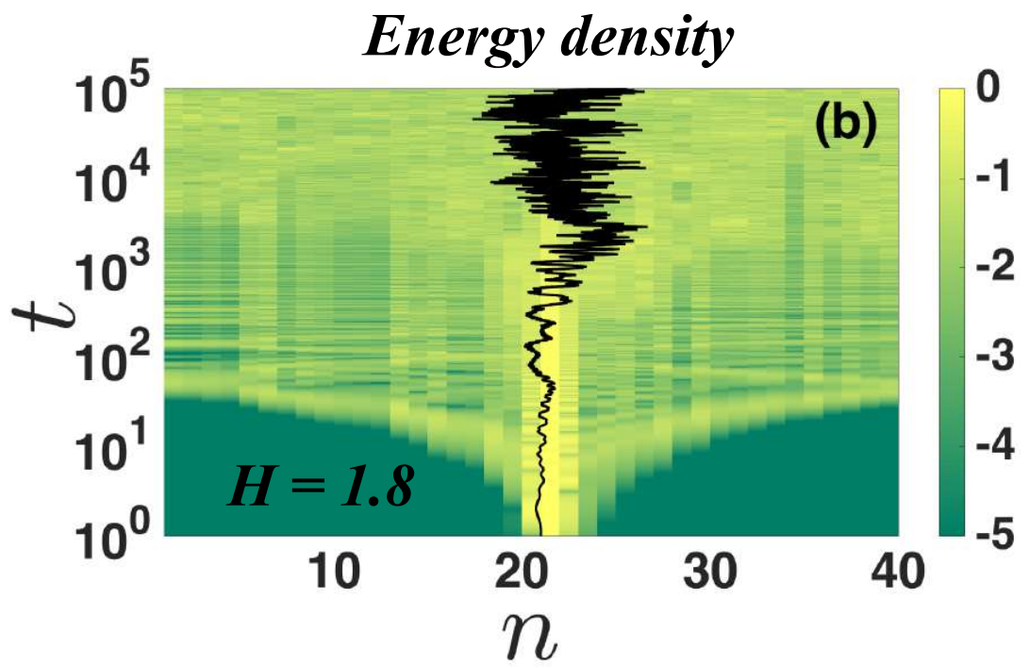
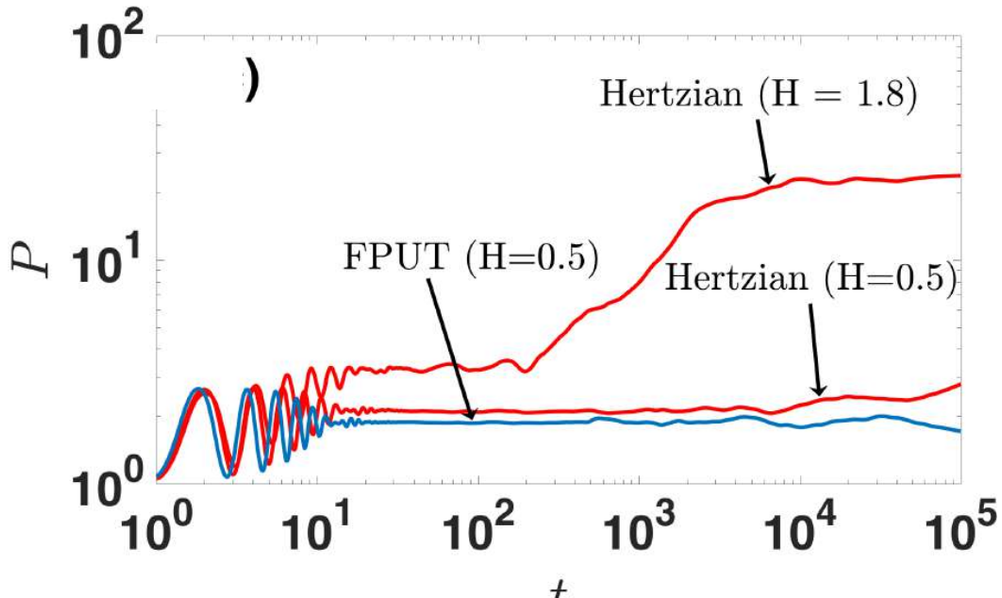
Hertzian model

$H = 0.5$ $H = 1.8$

Increasing energy







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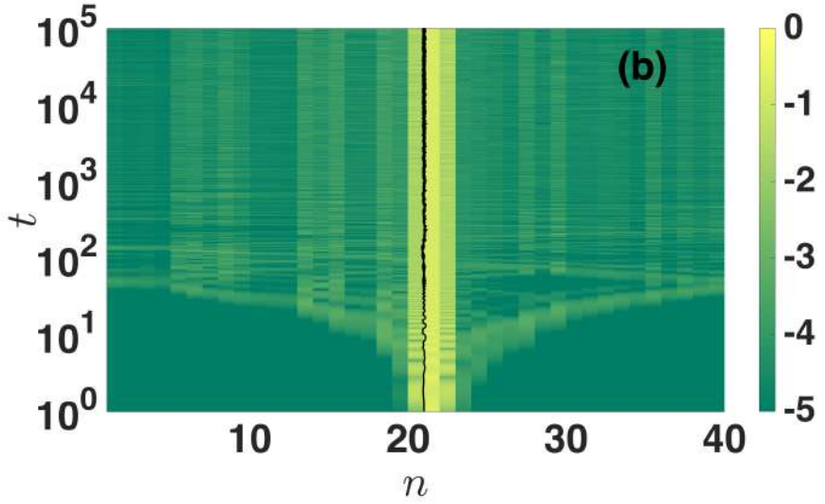
FPUT model

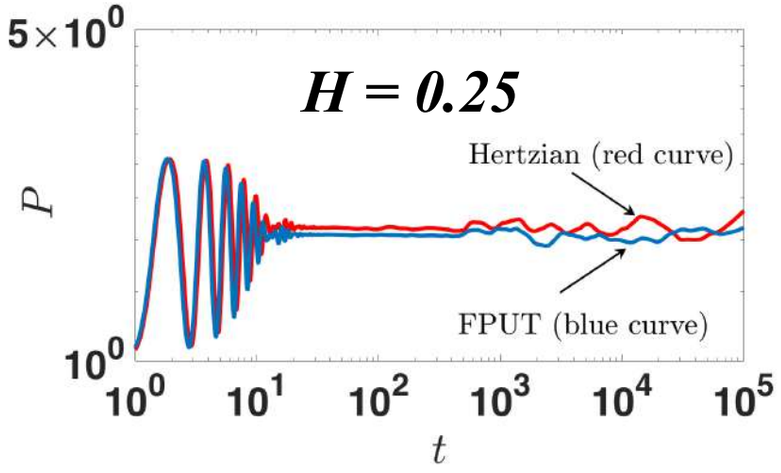
$H = 0.25$ $H = 2.9$

Increasing energy

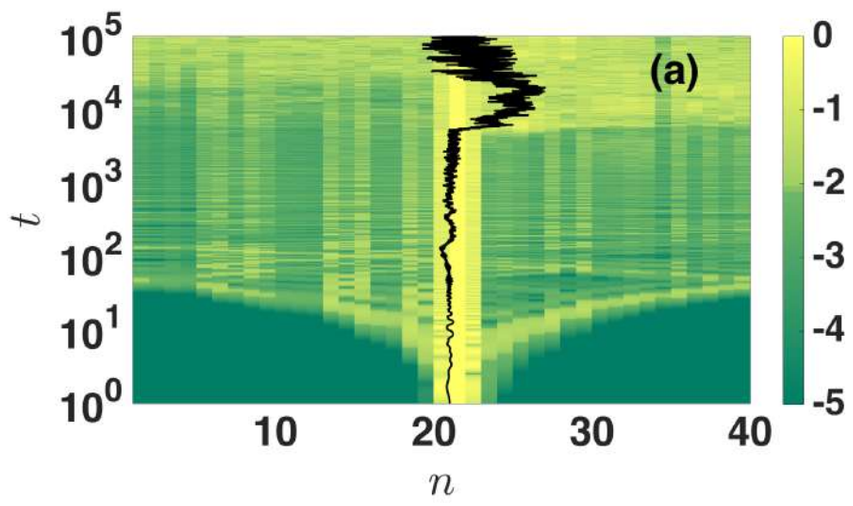
Energy density

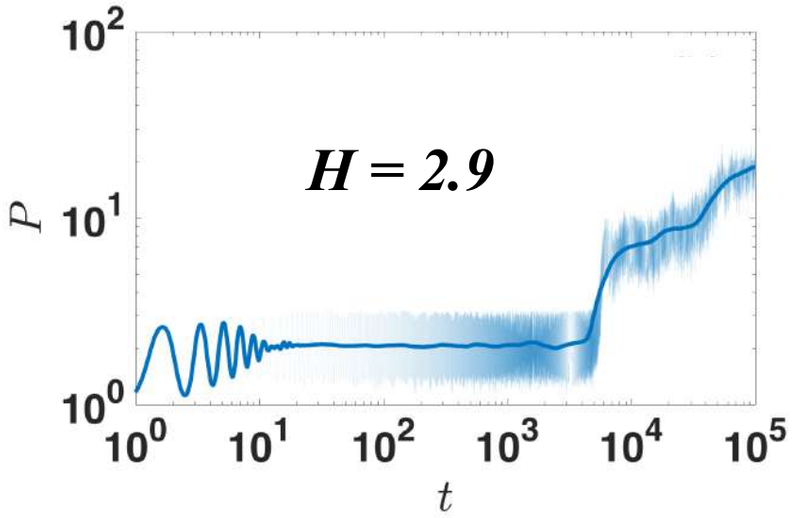


Participation number



Energy localization





Energy delocalization

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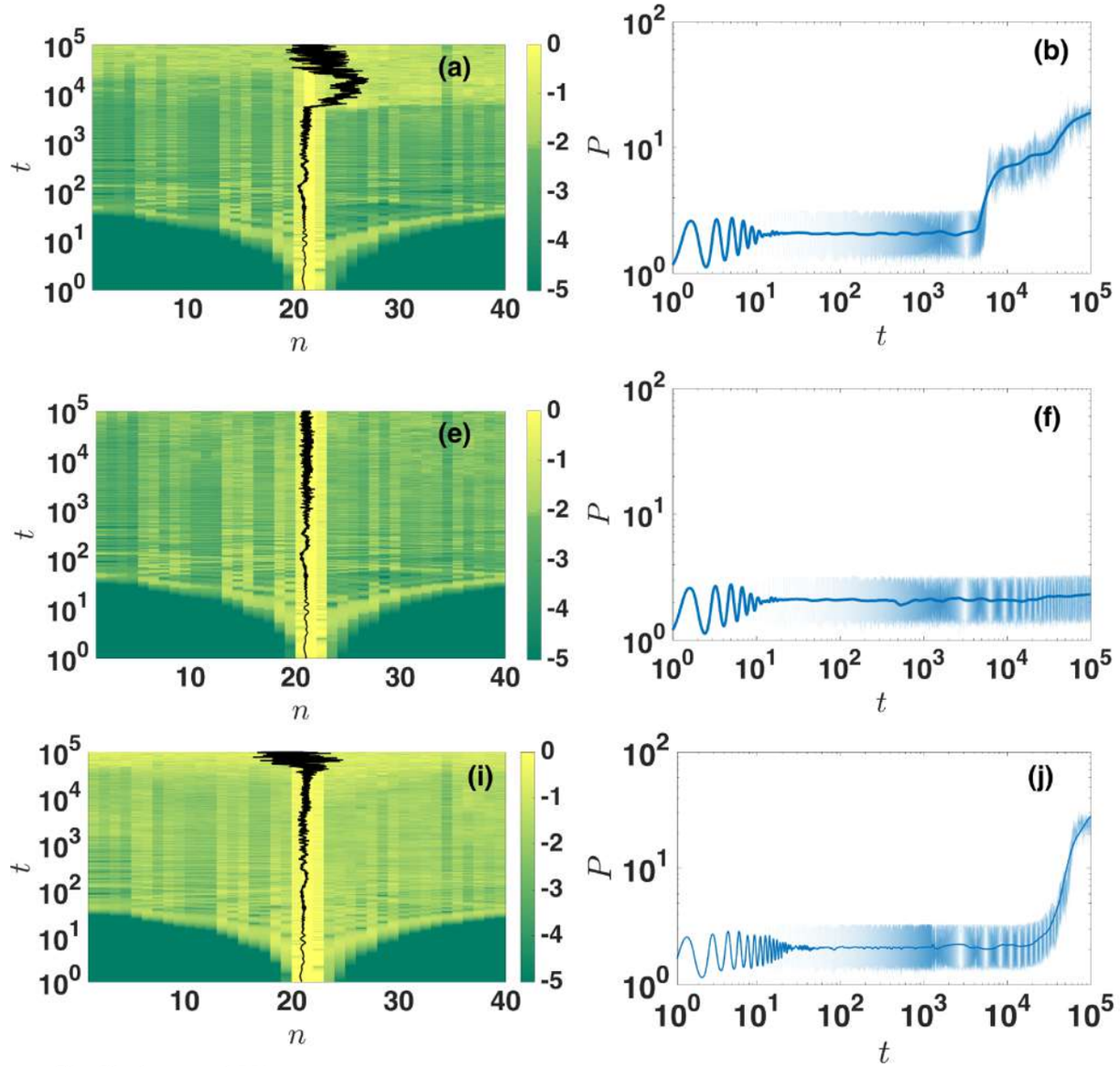
FPUT model

$H = 2.9$ $H = 4$ $H = 8$

Increasing energy

Increasing amplitude (nonlinearity)



still open
to understand why!

A. Ngapasare, G. Theocharis, O. Richoux, Ch. Skokos, and VA
Phys. Rev. E **99**, 032211 (2019)

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Energy equipartition Hertzian

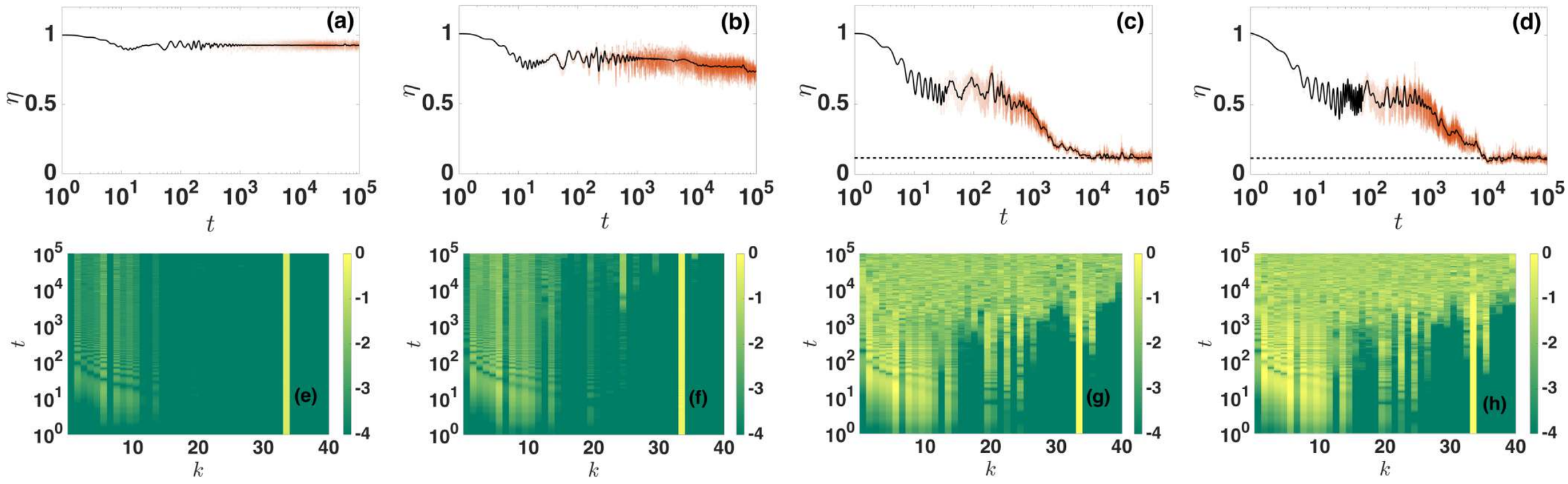
$$S(t) = -\sum_{k=1}^{N} v_k(t) \ln v_k(t). \qquad v_k = E_k / \sum_{k=1}^{N} E_k \qquad \eta(t) = \frac{S(t) - S_{max}}{S(0) - S_{max}}$$

$$\eta = 1$$

(All energy is on a single mode)

$$\eta = 0$$

(Energy is distributed equally amongst the modes)



Increasing amplitude (nonlinearity)

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Energy equipartition
FPU

$$S(t) = -\sum_{k=1}^{N} v_k(t) \ln v_k(t). \qquad v_k = E_k / \sum_{k=1}^{N} E_k \qquad \eta(t) = \frac{S(t) - S_{max}}{S(0) - S_{max}}$$

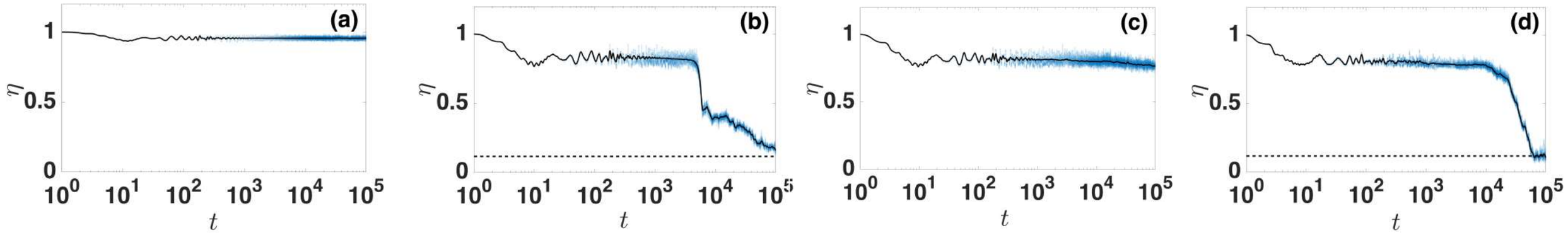
$$\eta = 1$$

(All energy is on a single mode)

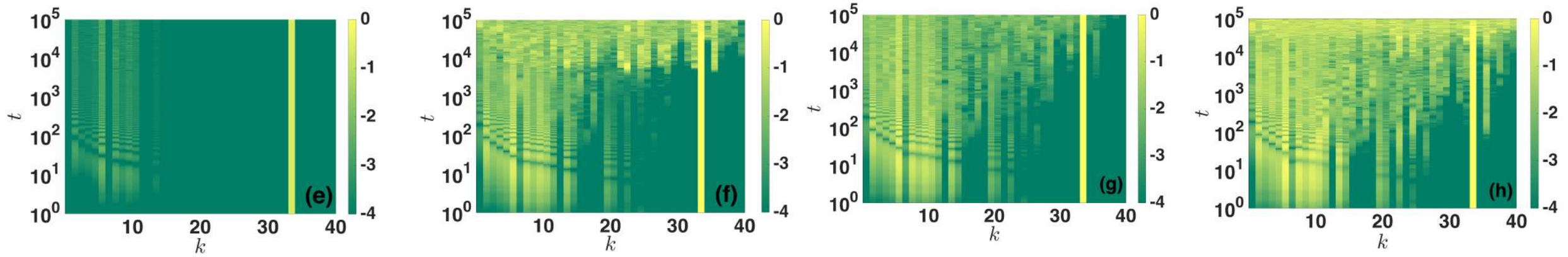
$$\eta = 0$$

(Energy is distributed equally amongst the modes)

Entropy



Mode Energy



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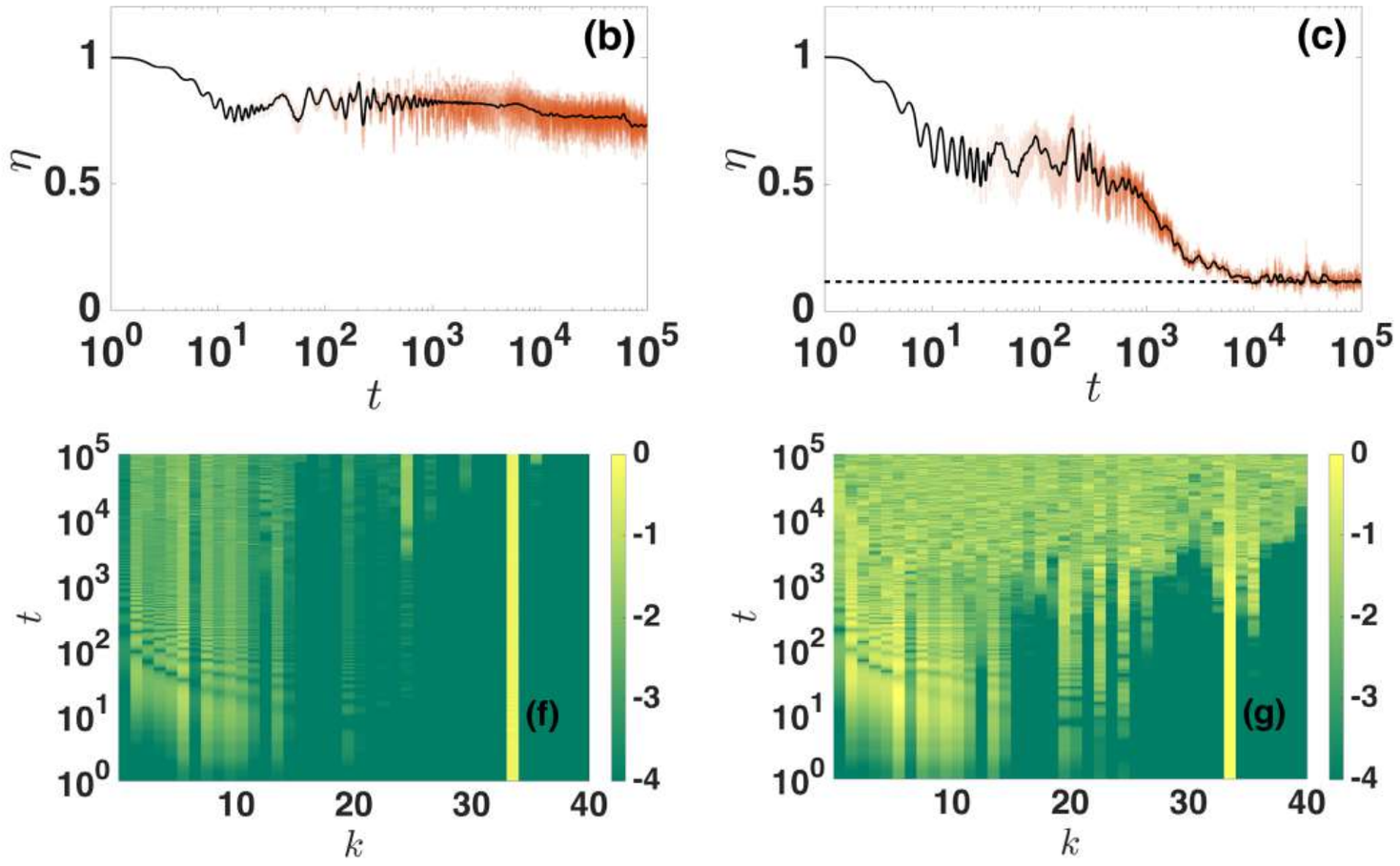
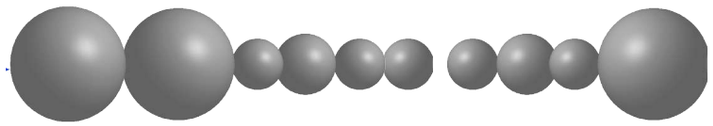
Exploring the role of discontinuity





normalized entropy

mode distribution

gaps in the lattice

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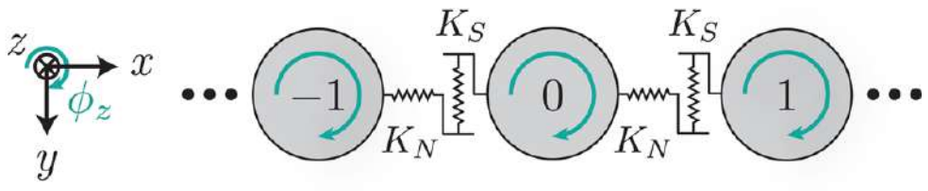
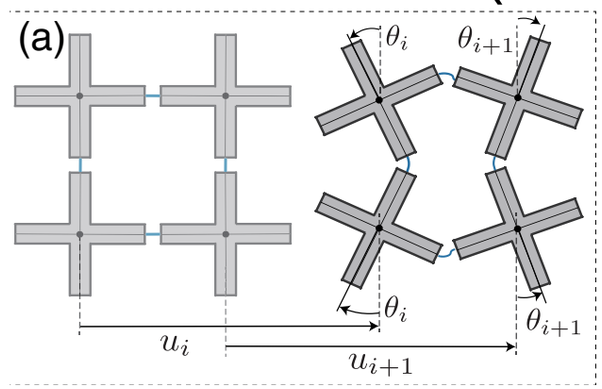
Including more degrees of freedom

Granular chains



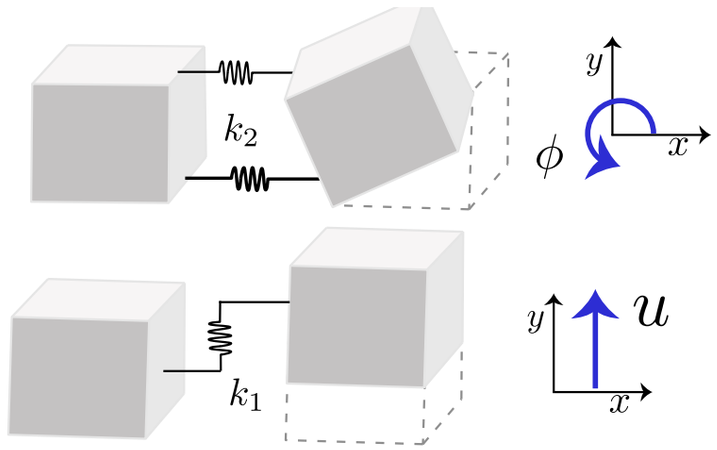
* F Allein, V Tournat, VE Gusev, G Theocharis, APL 108 (2016)

Elastic lattices (LEGO)



B. Deng, V. Tournat, P. Wang, and K. Bertoldi
Phys. Rev. Lett. **122**, 044101 (2019)

Simple model with rotations



A. A. Vasiliev, A. E. Miroshnichenko, M. Ruzzene, Mech. Res. Comm. **37** (2010)

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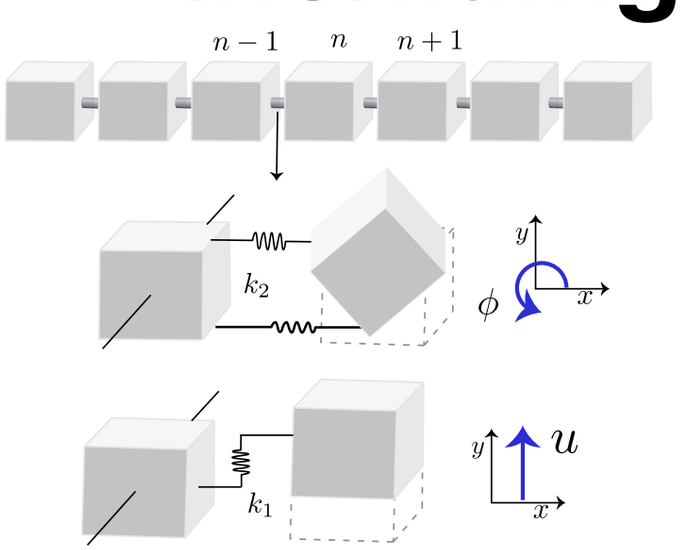
Including more degrees of freedom



$$m\ddot{u}_n = K_{n+1}^{(1)}\left(u_{n+1}-u_n\right) - K_n^{(1)}\left(u_n-u_{n-1}\right) + \frac{hK_{n+1}^{(1)}}{2}\left(\phi_{n+1}+\phi_n\right) - \frac{hK_n^{(1)}}{2}\left(\phi_n+\phi_{n-1}\right)$$

$$I\ddot{\phi}_n = \Big[\frac{hK_n^{(1)}}{2}\left(u_{n-1}-u_n\right) + \frac{hK_{n+1}^{(1)}}{2}\left(u_n-u_{n+1}\right) - \frac{h^2K_{n+1}^{(1)}}{4}\left(\phi_{n+1}+\phi_n\right) - \frac{h^2K_n^{(1)}}{4}\left(\phi_n+\phi_{n-1}\right)$$

$$+ K^{(2)}\left(\phi_{n+1}-\phi_n\right) - K^{(2)}\left(\phi_n-\phi_{n-1}\right)\Big]$$

Transverse *displacement* and *rotation*

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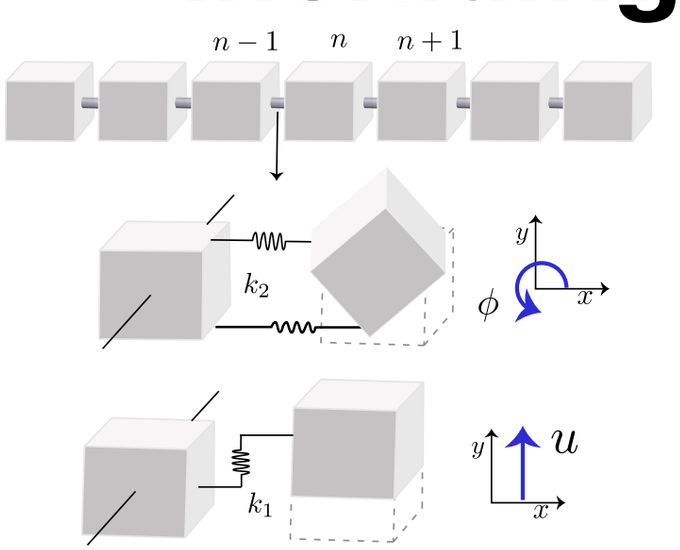
Including more degrees of freedom



$$m\ddot{u}_n = K_{n+1}^{(1)}\left(u_{n+1}-u_n\right) - K_n^{(1)}\left(u_n-u_{n-1}\right) + \frac{hK_{n+1}^{(1)}}{2}\left(\phi_{n+1}+\phi_n\right) - \frac{hK_n^{(1)}}{2}\left(\phi_n+\phi_{n-1}\right)$$

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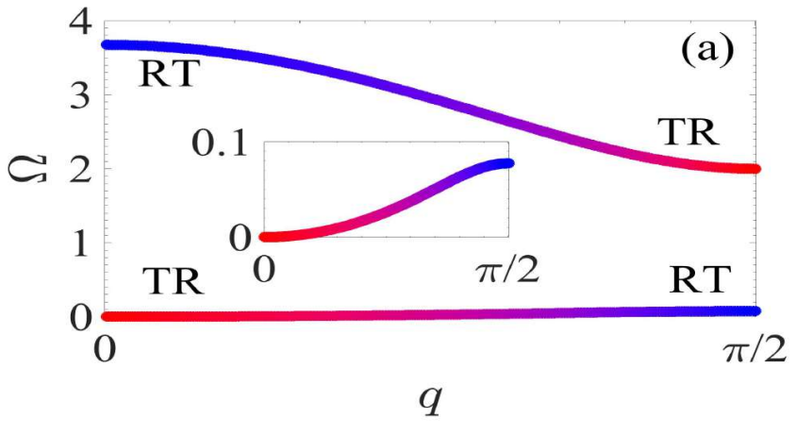
$$+ K^{(2)}\left(\phi_{n+1}-\phi_n\right) - K^{(2)}\left(\phi_n-\phi_{n-1}\right)\Big]$$

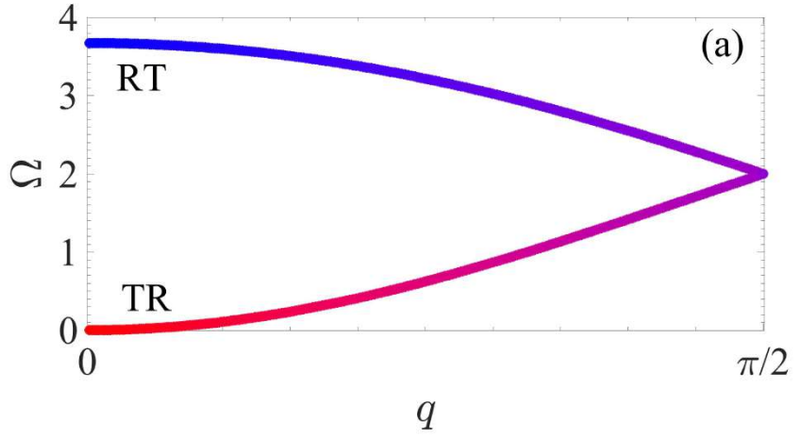
Transverse *displacement* and *rotation*

$K_n^{(1)} = 1$

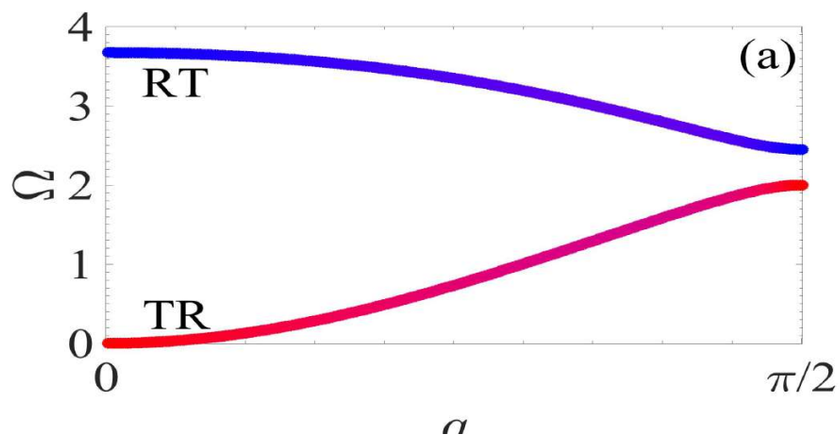
$$\Omega_\pm^2 = \frac{1}{2}\left\{\sin^2 q + p(\cos^2 q + p_B \sin^2 q) \pm \sqrt{\left[\sin^2 q + p(\cos^2 q + p_B \sin^2 q)\right]^2 - 4p_B p \sin^4 q}\right\}$$



$K^{(2)} = 10^{-3}$



$K^{(2)} = 2/3$



$K^{(2)} = 1$

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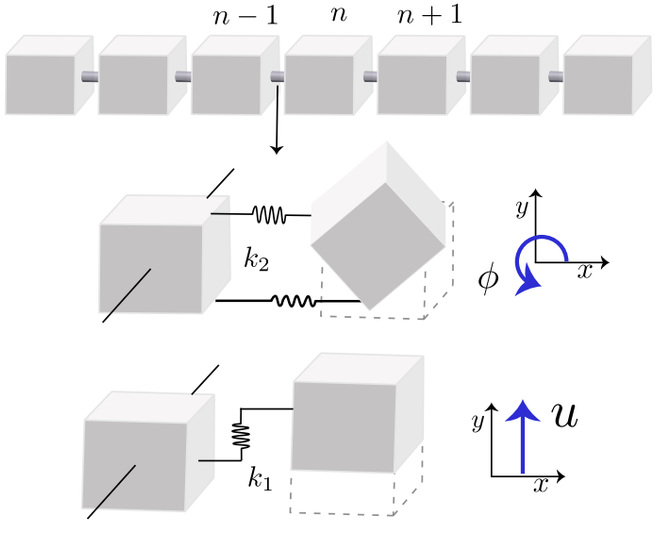
Strong disorder



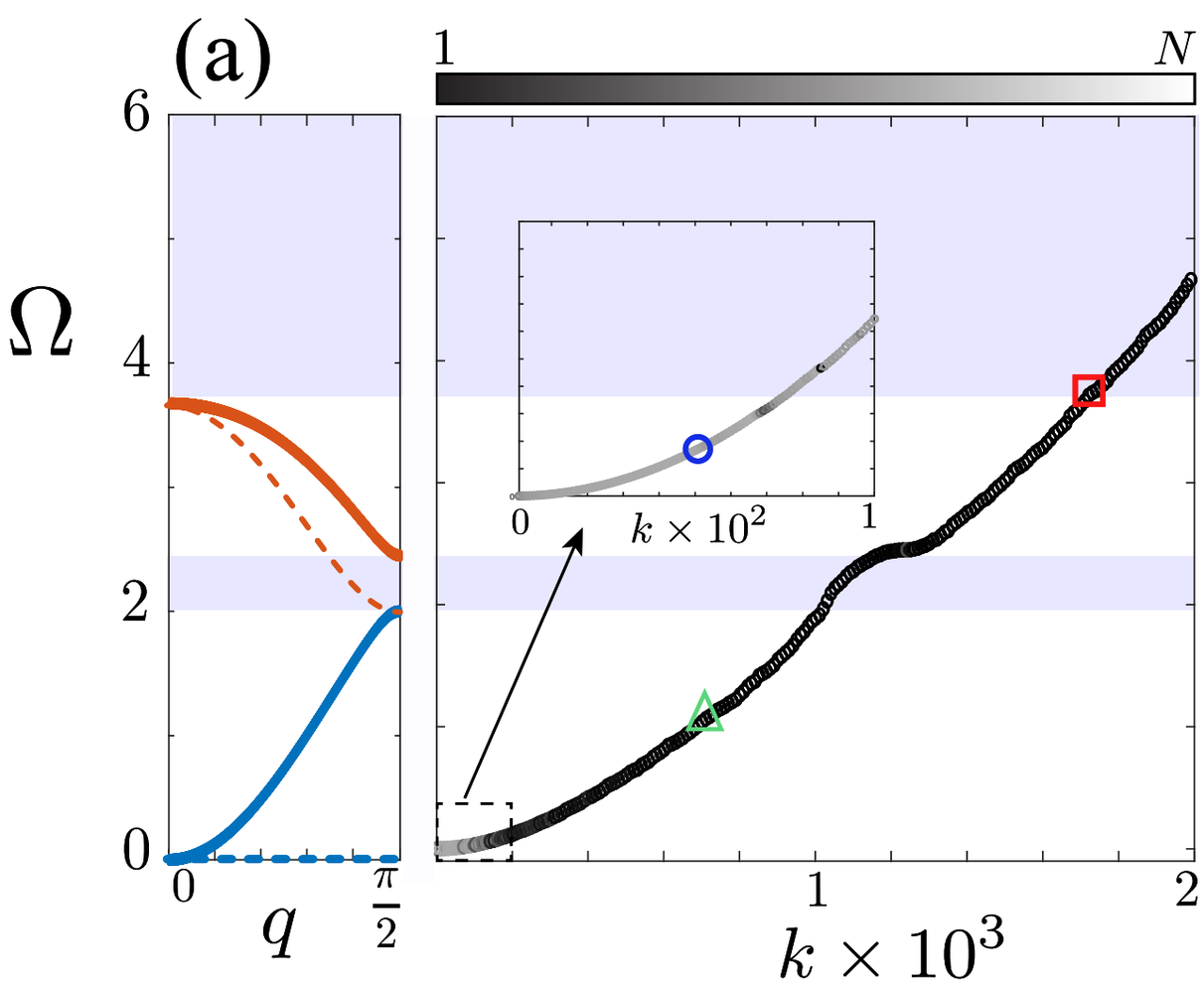
$$m\ddot{u}_n = K_{n+1}^{(1)}\left(u_{n+1}-u_n\right) - K_n^{(1)}\left(u_n-u_{n-1}\right) + \frac{hK_{n+1}^{(1)}}{2}\left(\phi_{n+1}+\phi_n\right) - \frac{hK_n^{(1)}}{2}\left(\phi_n+\phi_{n-1}\right)$$

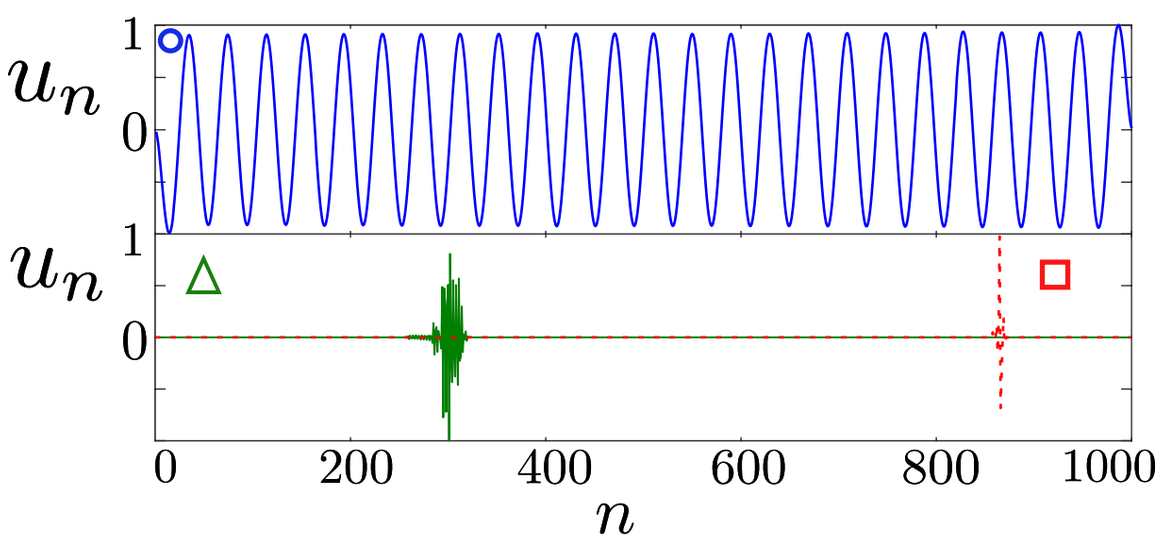
$$I\ddot{\phi}_n = \Big[\frac{hK_n^{(1)}}{2}\left(u_{n-1}-u_n\right) + \frac{hK_{n+1}^{(1)}}{2}\left(u_n-u_{n+1}\right) - \frac{h^2K_{n+1}^{(1)}}{4}\left(\phi_{n+1}+\phi_n\right) - \frac{h^2K_n^{(1)}}{4}\left(\phi_n+\phi_{n-1}\right)$$

$$+ K^{(2)}\left(\phi_{n+1}-\phi_n\right) - K^{(2)}\left(\phi_n-\phi_{n-1}\right)\Big]$$

$$f\left(K_n^{(1)}\right) = \begin{cases} W^{-1}, & -W/2 < \quad 1 \quad < W/2 \\ 0 & \text{otherwise.} \end{cases}$$

$$K^{(2)} = 1$$





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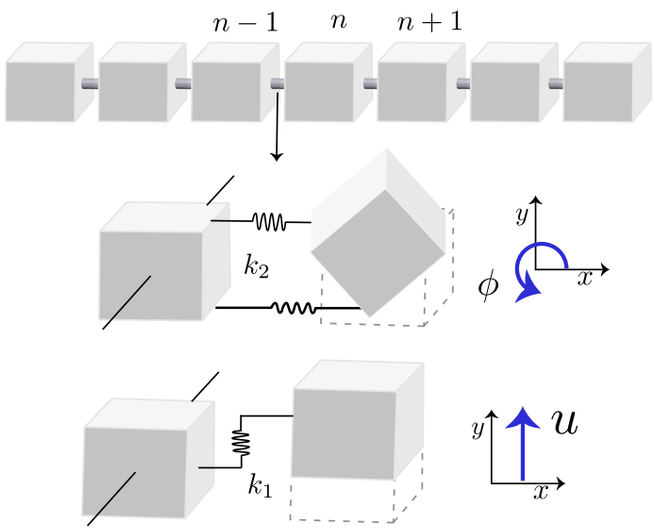
Strong disorder



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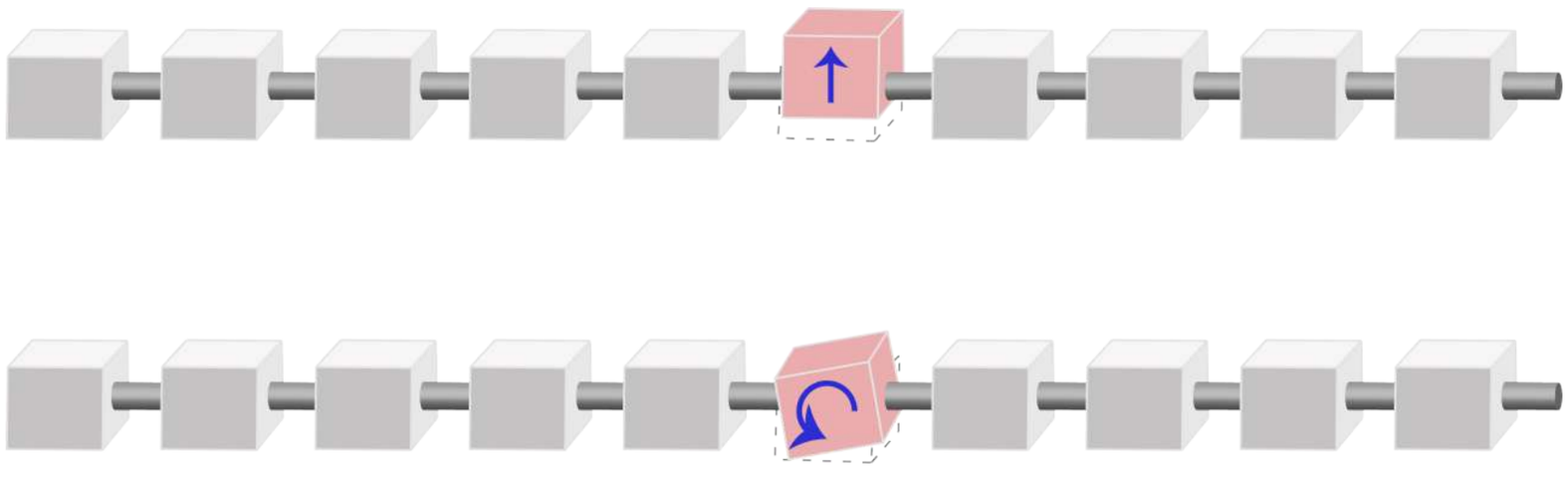
$$+ K^{(2)}\left(\phi_{n+1}-\phi_n\right) - K^{(2)}\left(\phi_n-\phi_{n-1}\right)\Big]$$

Transverse *displacement* and *rotation* - *Initial momenta*



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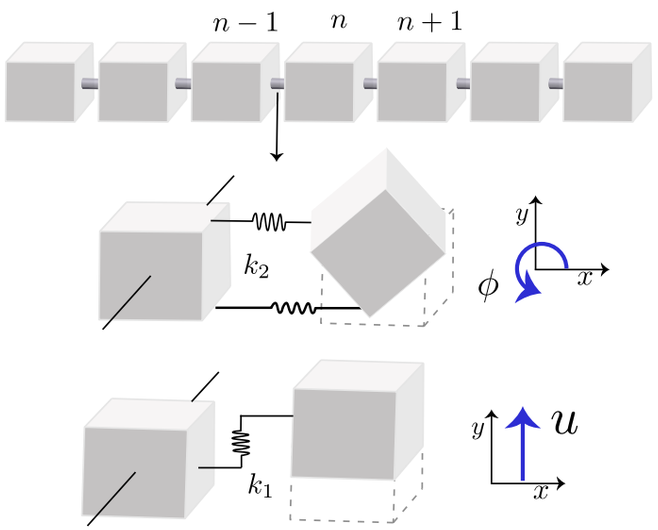
Strong disorder



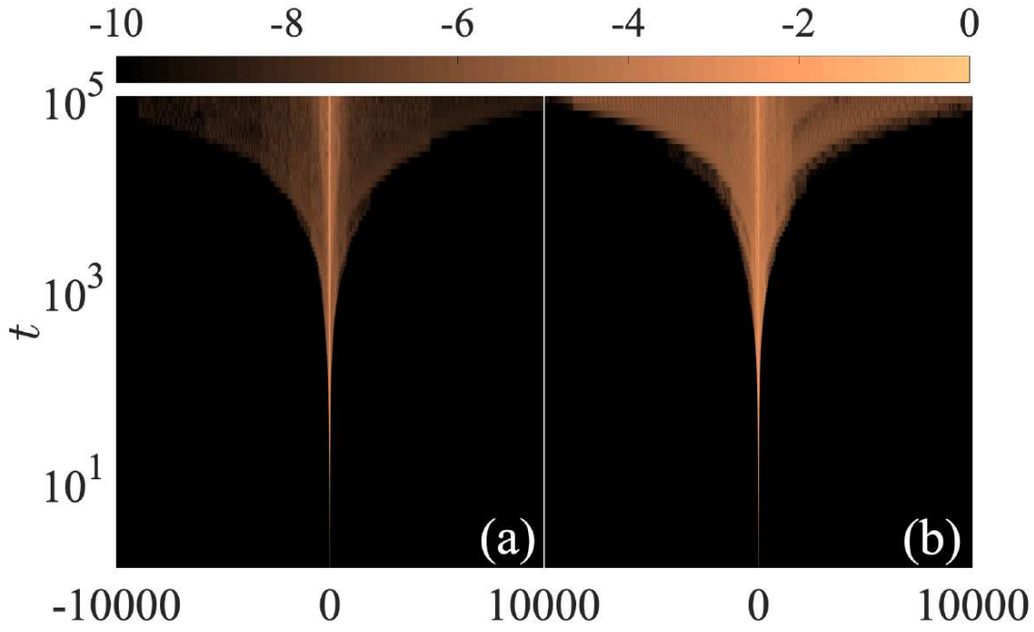
$$m\ddot{u}_n = K_{n+1}^{(1)}\left(u_{n+1}-u_n\right) - K_n^{(1)}\left(u_n-u_{n-1}\right) + \frac{hK_{n+1}^{(1)}}{2}\left(\phi_{n+1}+\phi_n\right) - \frac{hK_n^{(1)}}{2}\left(\phi_n+\phi_{n-1}\right)$$

$$I\ddot{\phi}_n = \Big[\frac{hK_n^{(1)}}{2}\left(u_{n-1}-u_n\right) + \frac{hK_{n+1}^{(1)}}{2}\left(u_n-u_{n+1}\right) - \frac{h^2K_{n+1}^{(1)}}{4}\left(\phi_{n+1}+\phi_n\right) - \frac{h^2K_n^{(1)}}{4}\left(\phi_n+\phi_{n-1}\right)$$

$$+ K^{(2)}\left(\phi_{n+1}-\phi_n\right) - K^{(2)}\left(\phi_n-\phi_{n-1}\right)\Big]$$

$$K^{(2)} = 1$$

Transverse *displacement* and *rotation* - *Initial momenta*



$$P_n^{(u)}(0) = \delta_{n0} \qquad P_n^{(\phi)}(0) = \sqrt{I}\delta_{n0}$$

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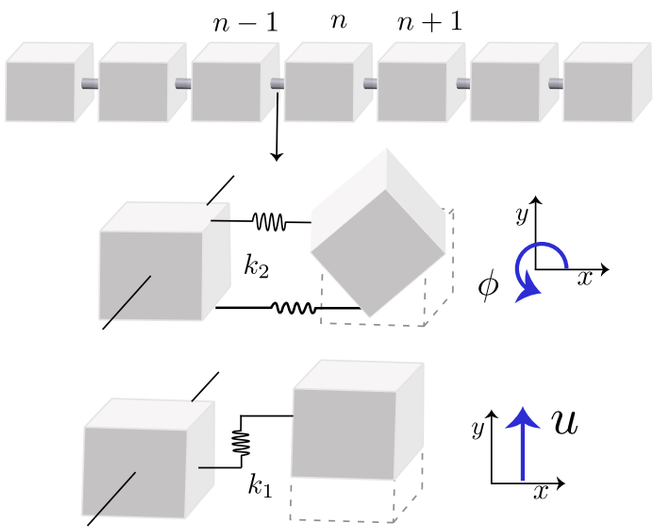
Strong disorder



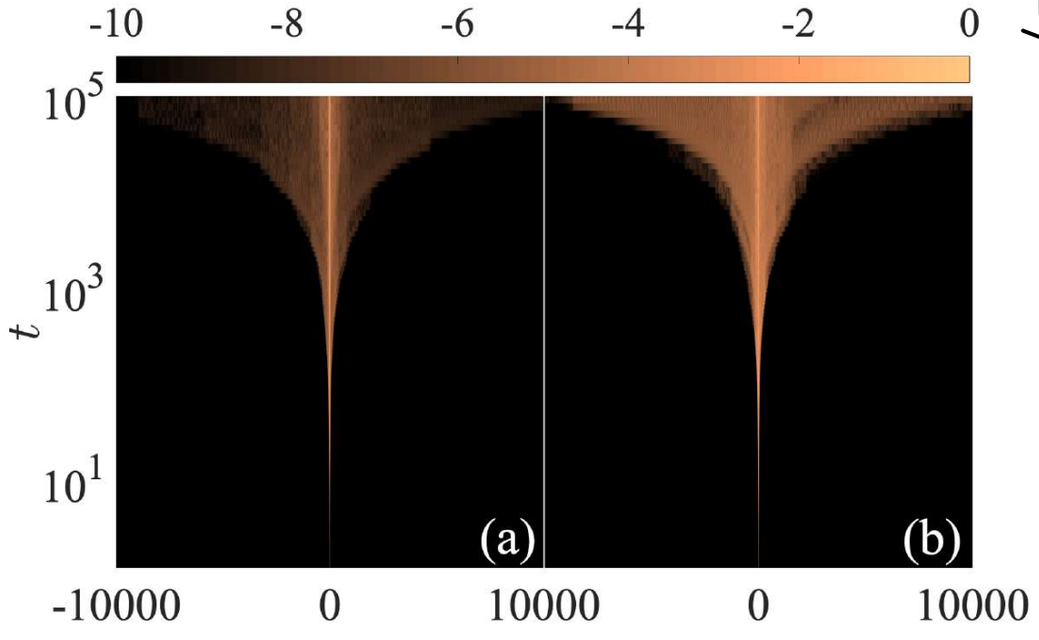
$$m\ddot{u}_n = K_{n+1}^{(1)}\left(u_{n+1}-u_n\right) - K_n^{(1)}\left(u_n-u_{n-1}\right) + \frac{hK_{n+1}^{(1)}}{2}\left(\phi_{n+1}+\phi_n\right) - \frac{hK_n^{(1)}}{2}\left(\phi_n+\phi_{n-1}\right)$$

$$I\ddot{\phi}_n = \Big[\frac{hK_n^{(1)}}{2}\left(u_{n-1}-u_n\right) + \frac{hK_{n+1}^{(1)}}{2}\left(u_n-u_{n+1}\right) - \frac{h^2K_{n+1}^{(1)}}{4}\left(\phi_{n+1}+\phi_n\right) - \frac{h^2K_n^{(1)}}{4}\left(\phi_n+\phi_{n-1}\right)$$

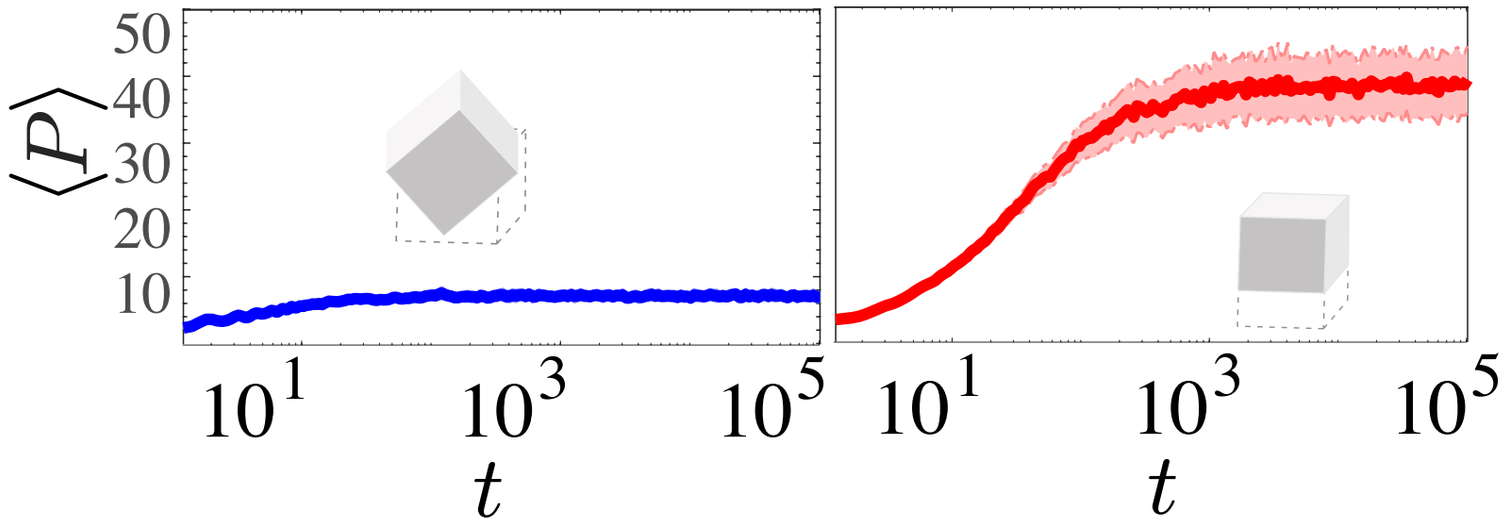
$$+ K^{(2)}\left(\phi_{n+1}-\phi_n\right) - K^{(2)}\left(\phi_n-\phi_{n-1}\right)\Big]$$

$$K^{(2)} = 1$$

Transverse *displacement* and *rotation* - *Initial momenta*



$$P_n^{(u)}(0) = \delta_{n0} \qquad P_n^{(\phi)}(0) = \sqrt{I}\delta_{n0}$$



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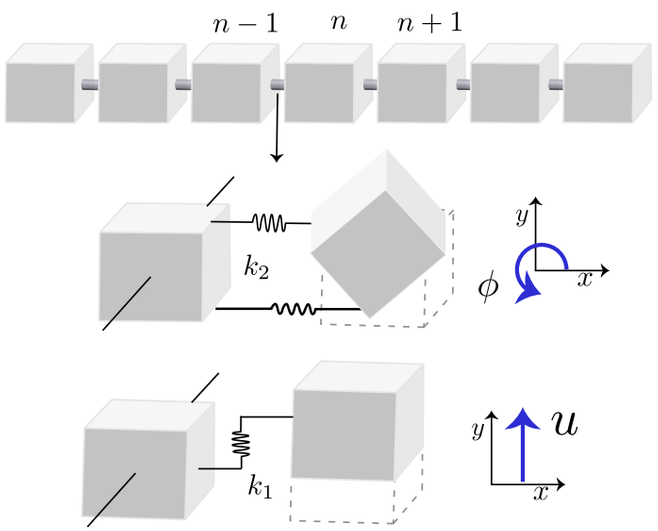
Strong disorder



$$m\ddot{u}_n = K_{n+1}^{(1)}\left(u_{n+1}-u_n\right) - K_n^{(1)}\left(u_n-u_{n-1}\right) + \frac{hK_{n+1}^{(1)}}{2}\left(\phi_{n+1}+\phi_n\right) - \frac{hK_n^{(1)}}{2}\left(\phi_n+\phi_{n-1}\right)$$

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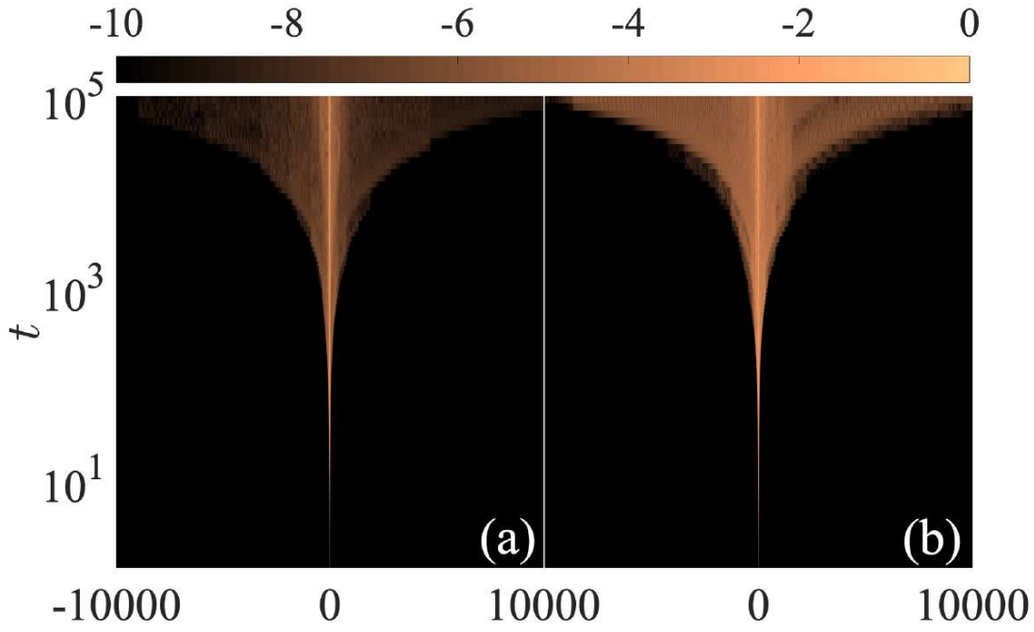
$$K^{(2)} = 1$$

Transverse *displacement* and *rotation* - *Initial momenta*



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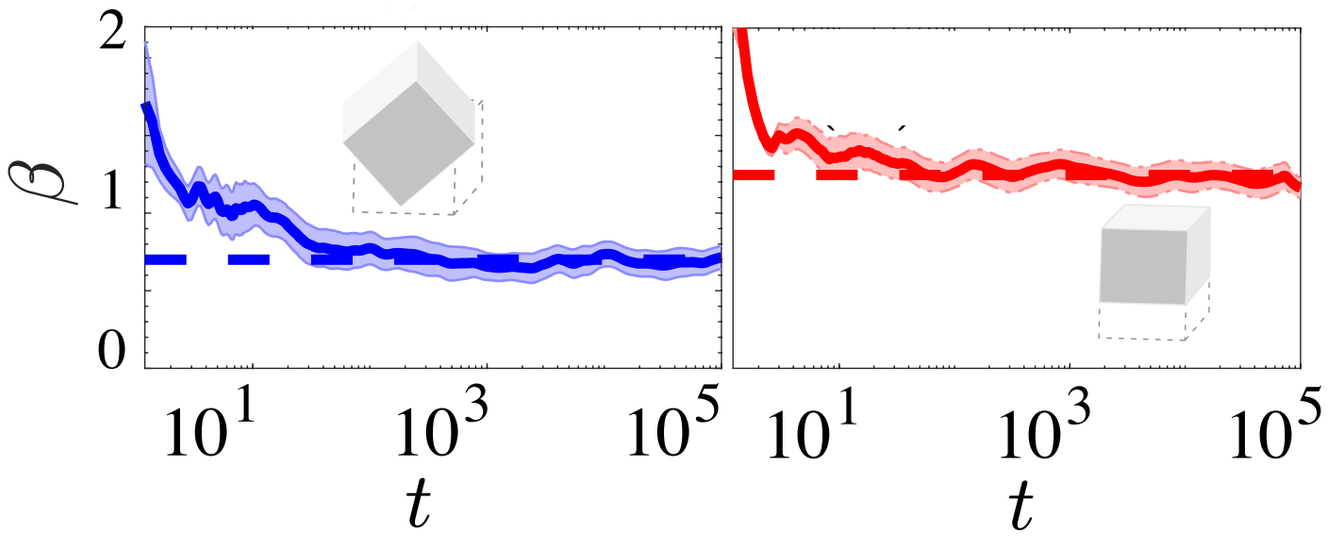

$\beta \approx 0.75$ subdiffusive

$\beta \approx 1.25$ superdiffusive

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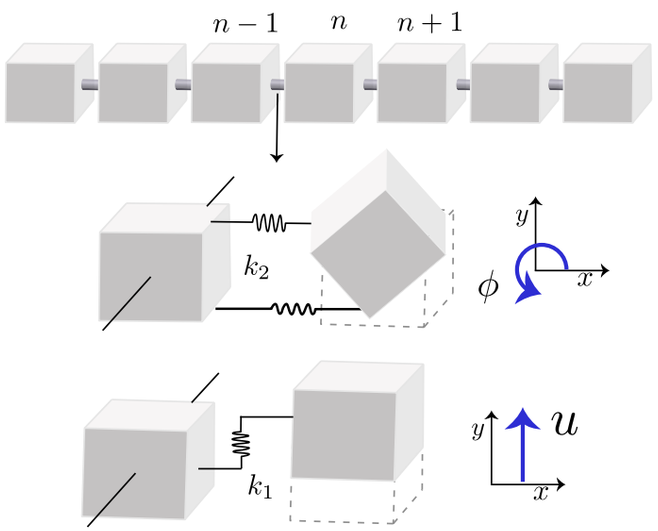
Strong disorder



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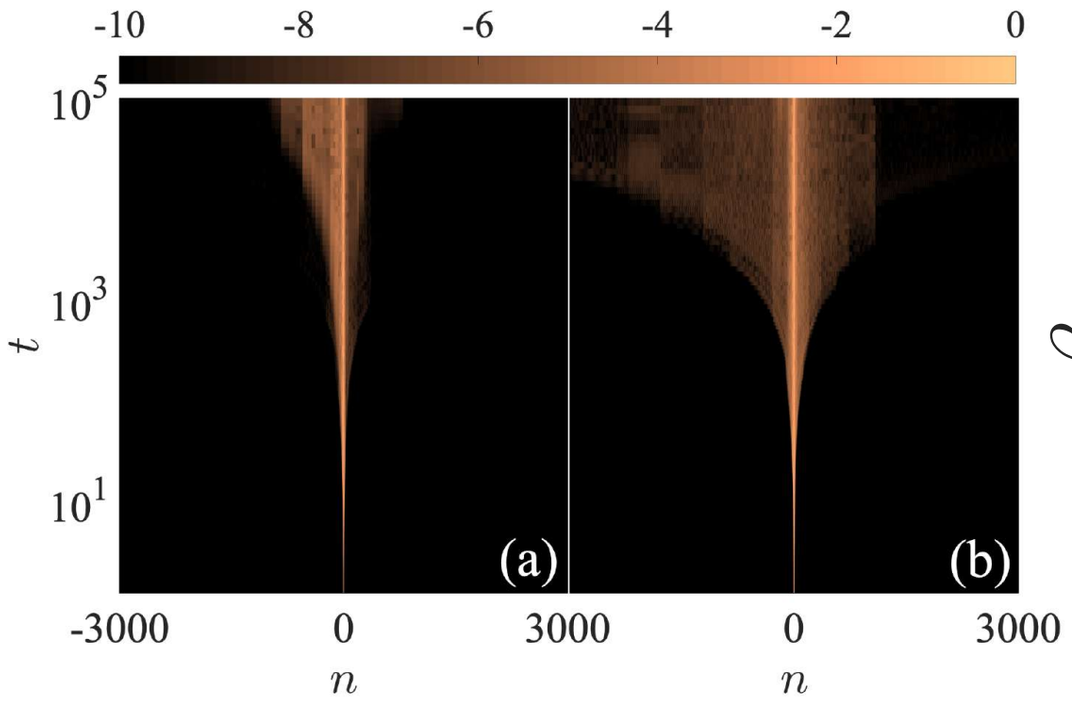
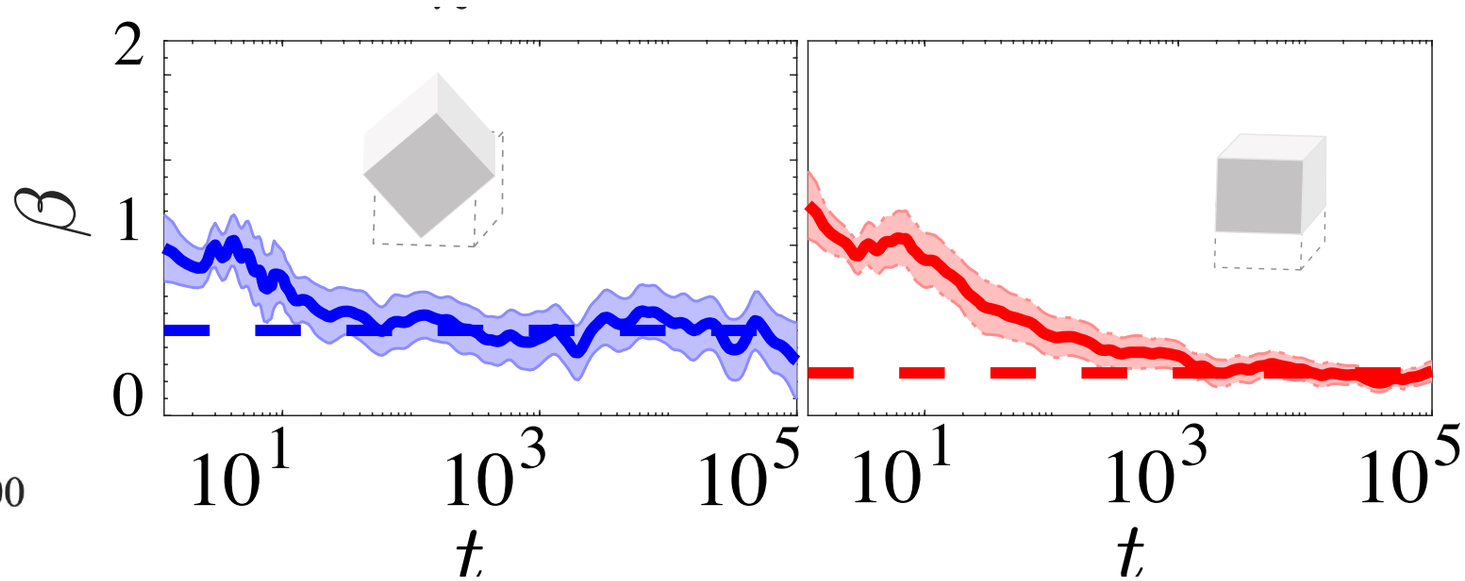
Transverse *displacement* and *rotation* - *Initial displacement*



$$\phi_n(0) = \phi_0\delta_{n0} \qquad u_n(0) = u_0\delta_{n0}$$



$\beta \approx 0.5$ subdiffusive

$\beta \approx 0.25$ subdiffusive

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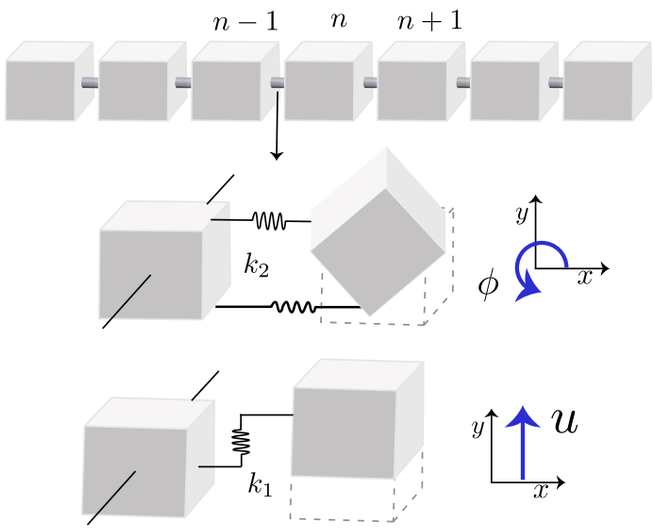
Strong disorder



$$m\ddot{u}_n = K_{n+1}^{(1)}\left(u_{n+1}-u_n\right) - K_n^{(1)}\left(u_n-u_{n-1}\right) + \frac{hK_{n+1}^{(1)}}{2}\left(\phi_{n+1}+\phi_n\right) - \frac{hK_n^{(1)}}{2}\left(\phi_n+\phi_{n-1}\right)$$

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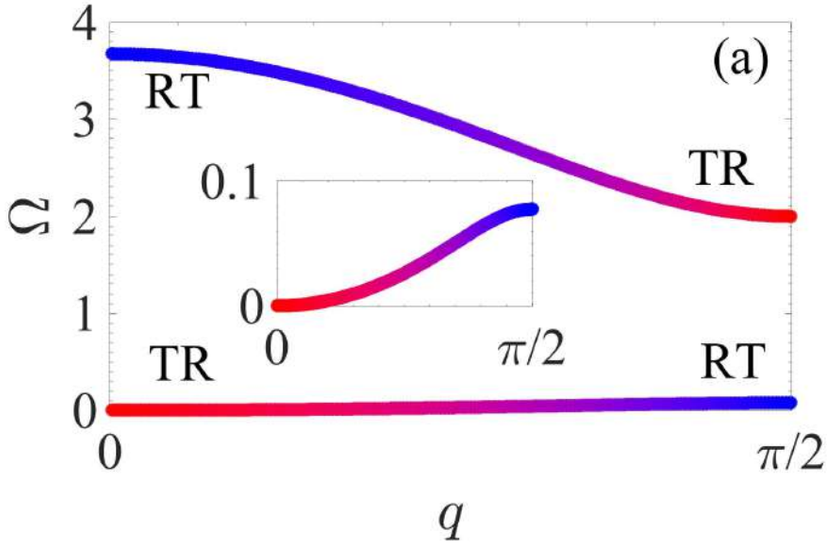
$$K^{(2)} = 10^{-3}$$

Can we tune the dynamics using the bending stiffness (coupling)



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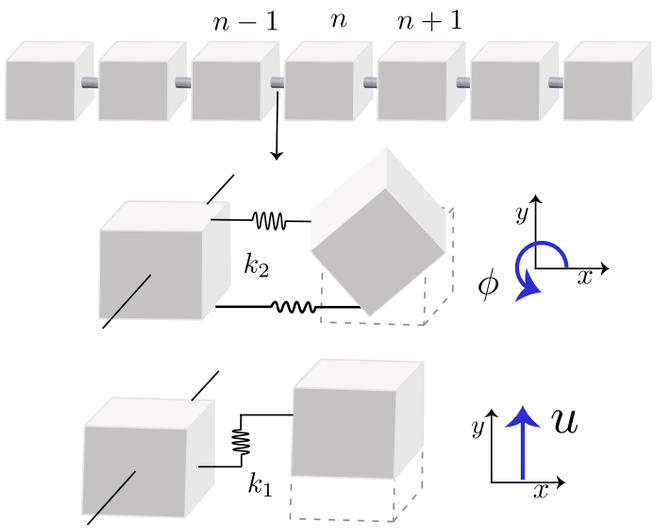
Strong disorder



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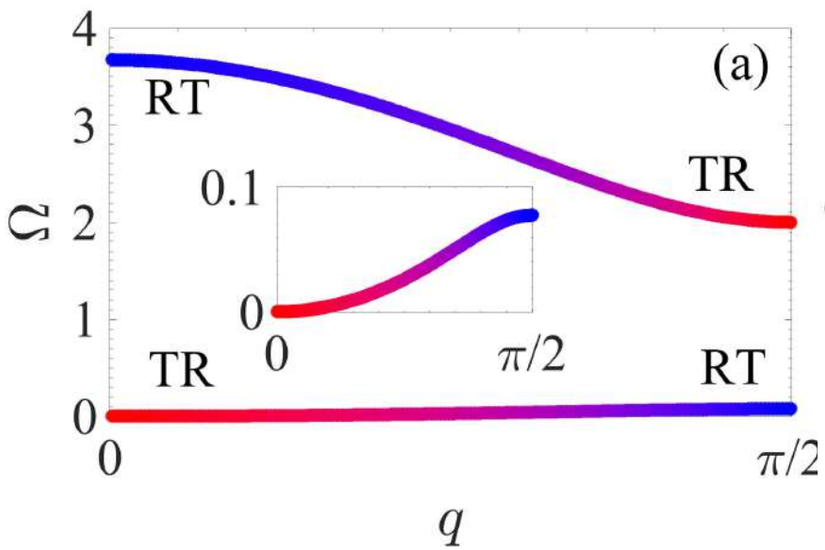

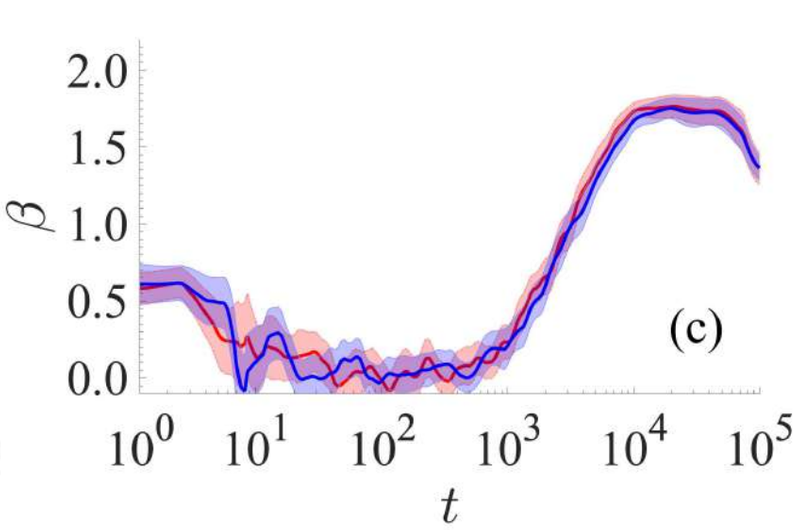

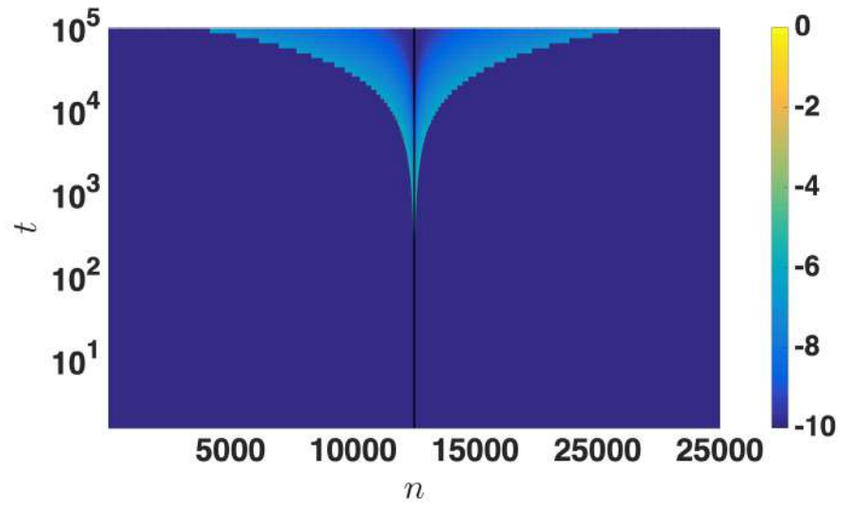

Transient dynamics due to the slow propagating modes

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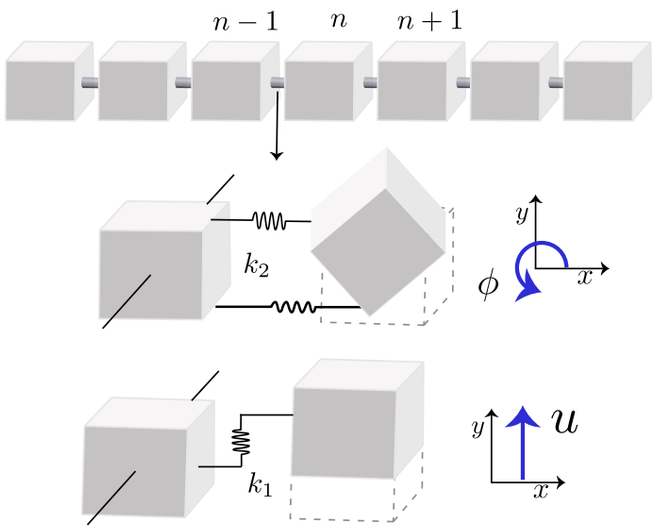
Strong disorder



$$m\ddot{u}_n = K_{n+1}^{(1)}\left(u_{n+1}-u_n\right) - K_n^{(1)}\left(u_n-u_{n-1}\right) + \frac{hK_{n+1}^{(1)}}{2}\left(\phi_{n+1}+\phi_n\right) - \frac{hK_n^{(1)}}{2}\left(\phi_n+\phi_{n-1}\right)$$

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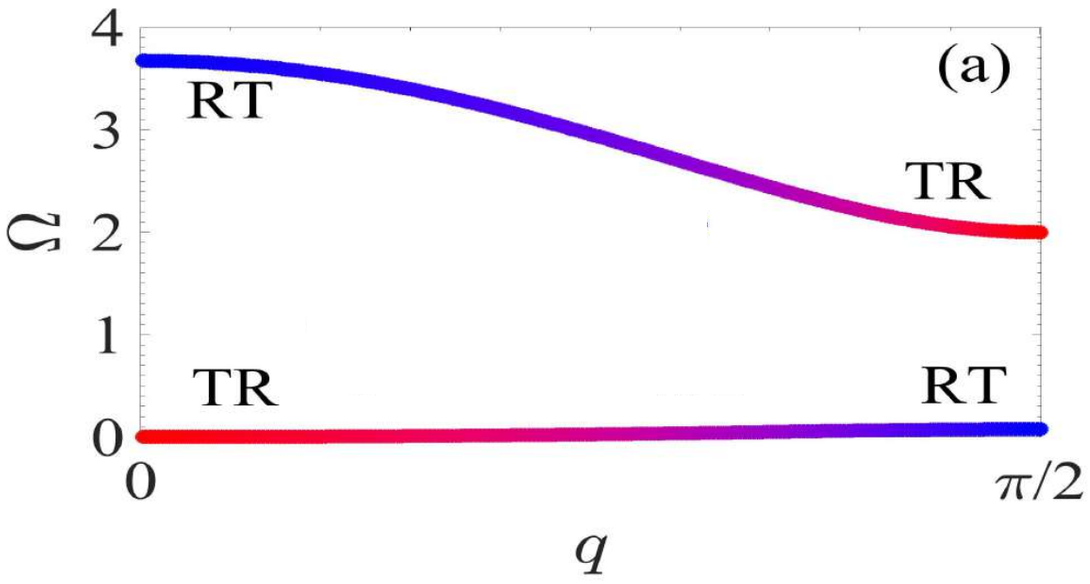
Can we tune the dynamics using the bending stiffness (coupling)



Soft modes due to the
balance between the two motions

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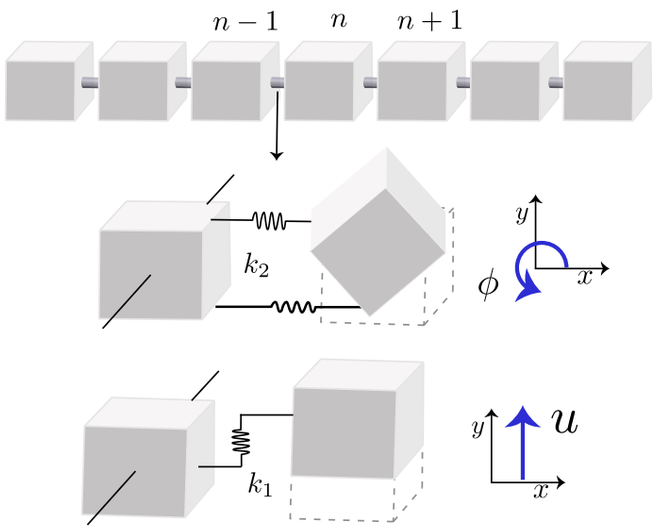
Strong disorder



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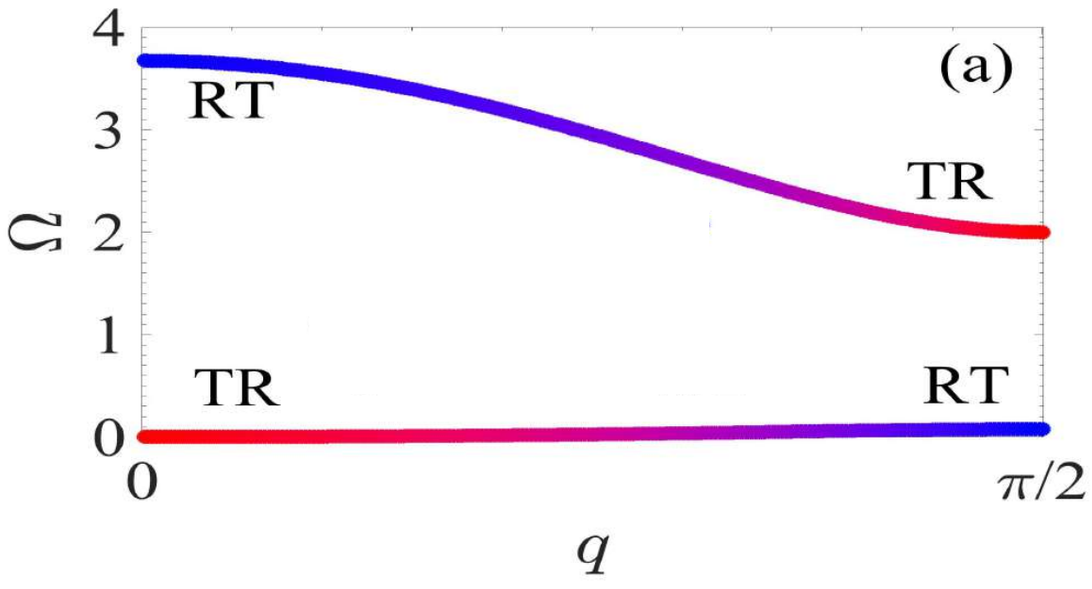
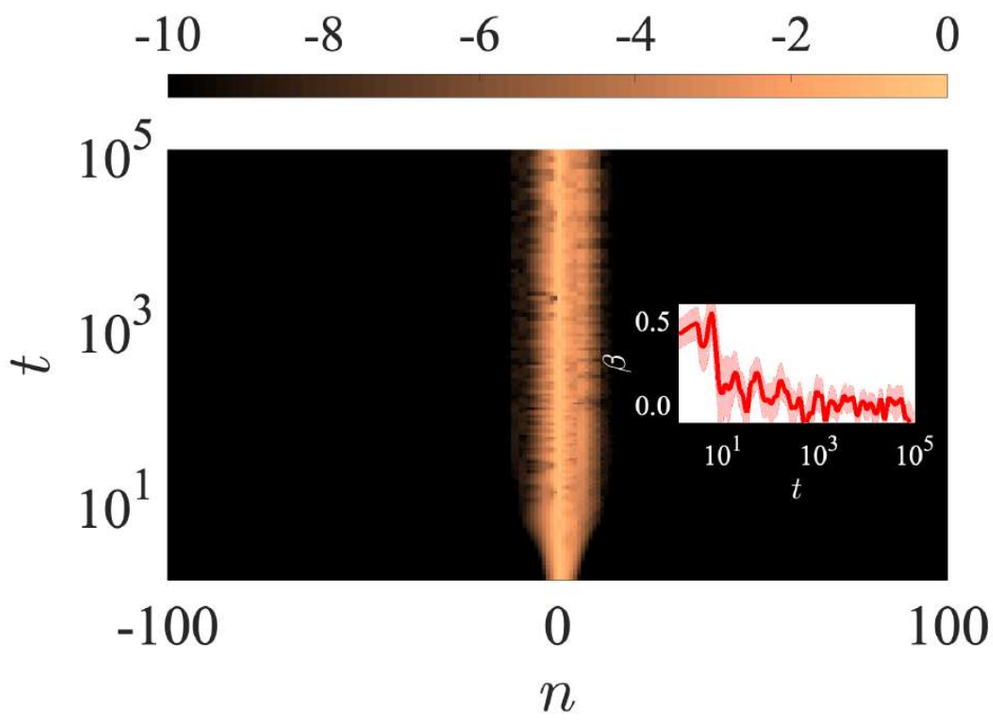


Soft modes due to the balance between the two motions



Behaves like a linear K.G.

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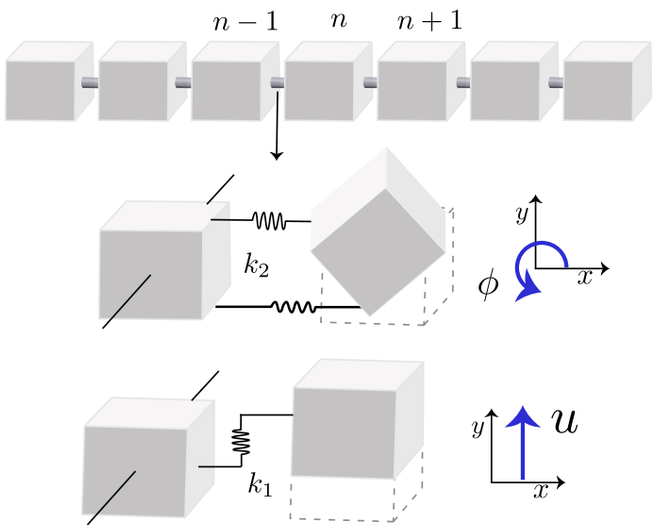
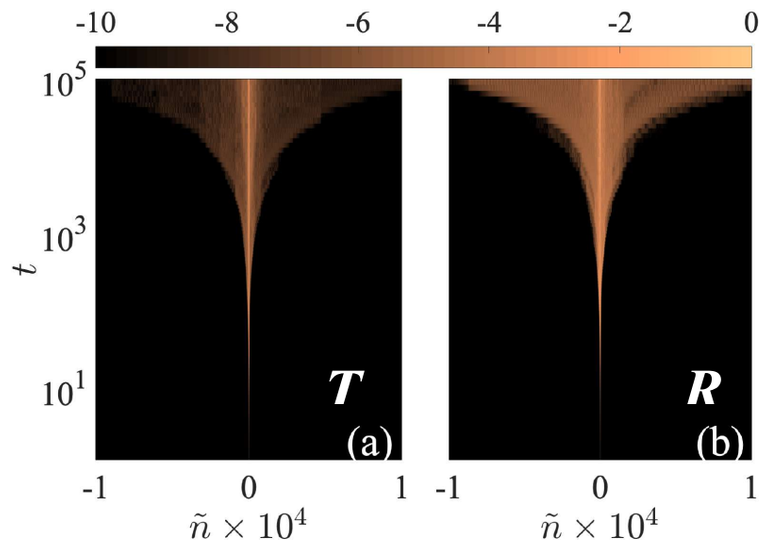
Polarisation





How energy is distributed in the center and the edges

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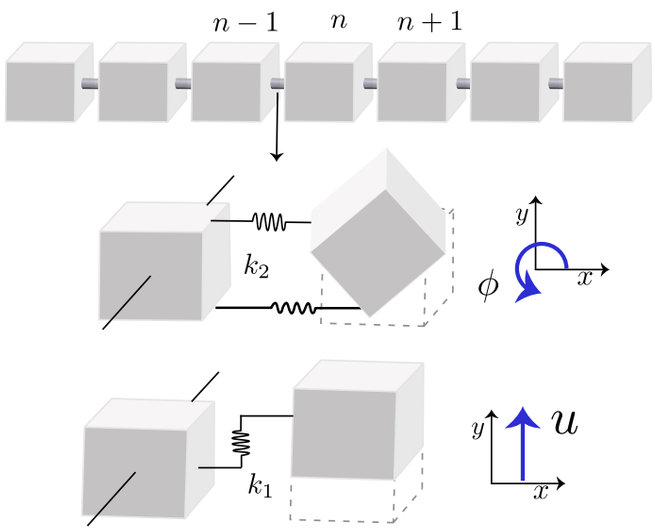
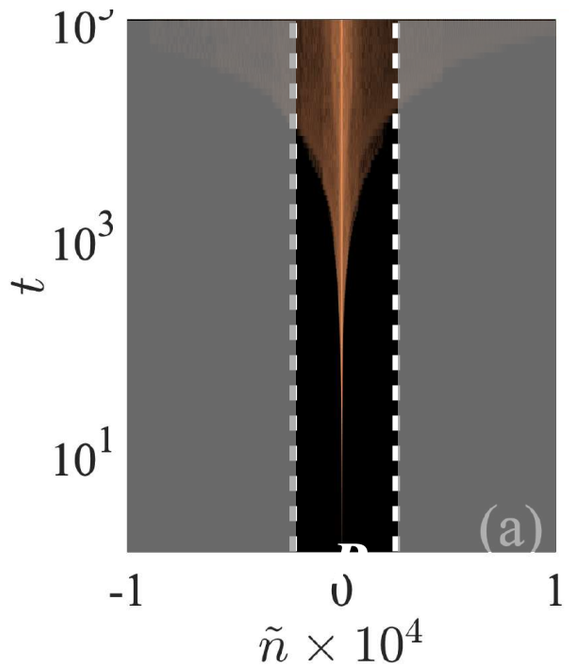
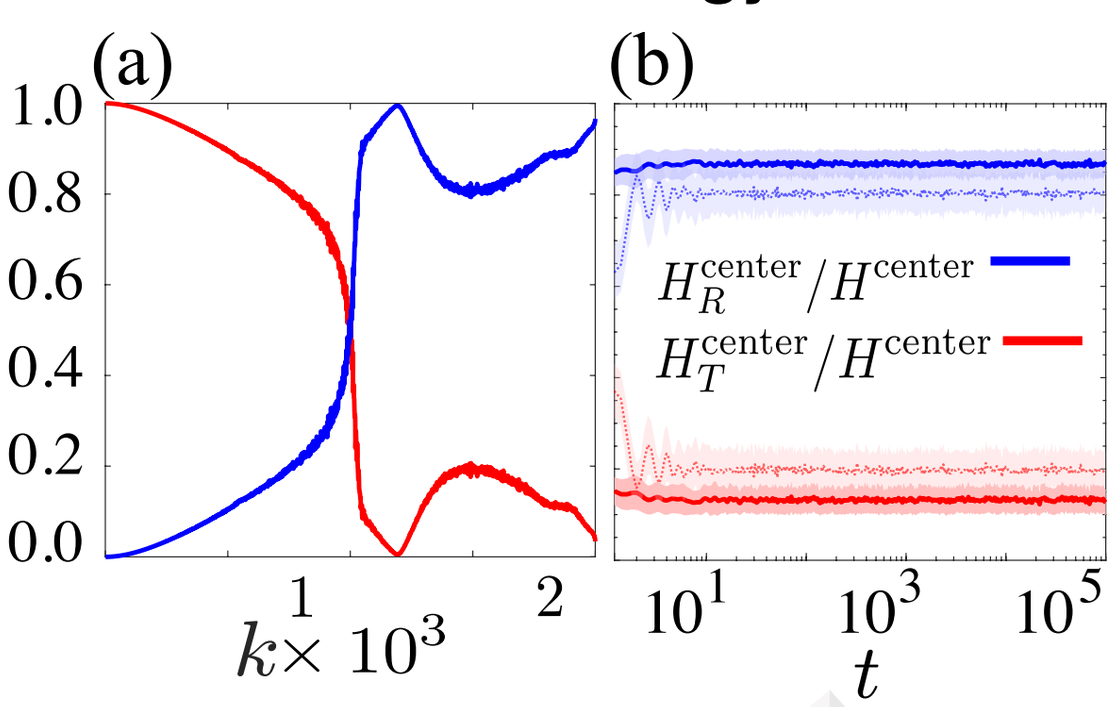
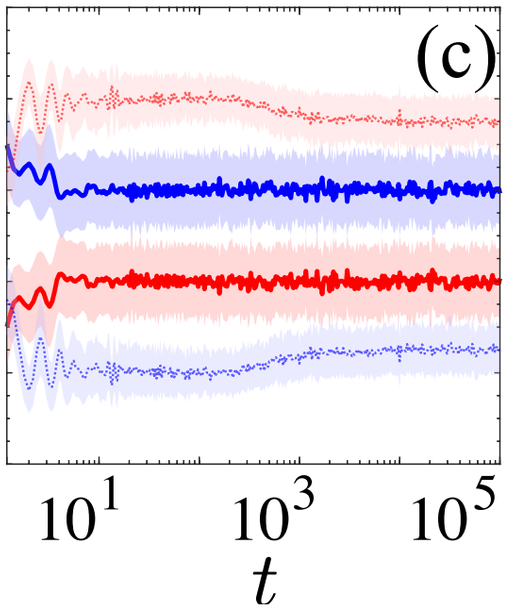
Polarisation

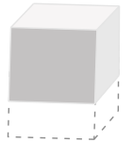




How energy is distributed in the <u>center</u> and the edges







Rotation

Transverse disp.

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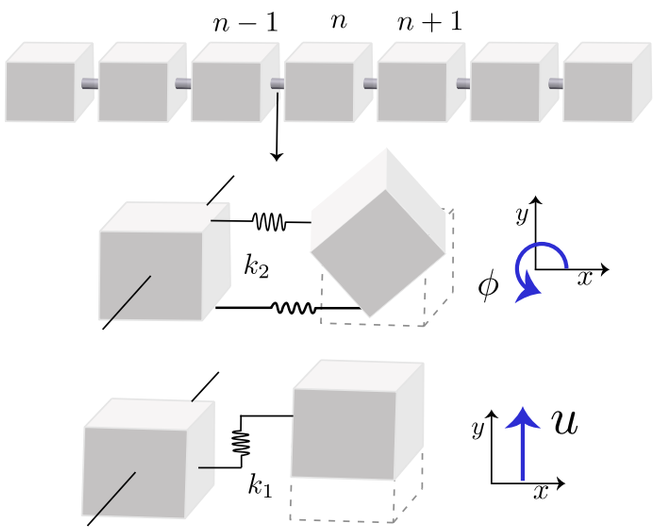
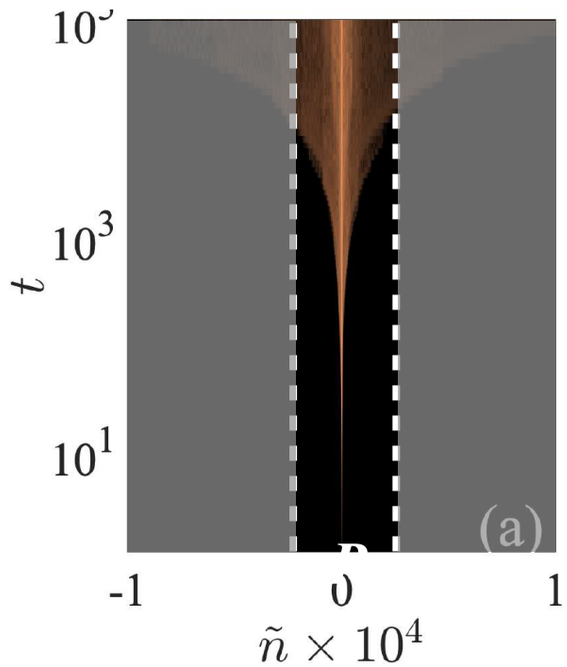
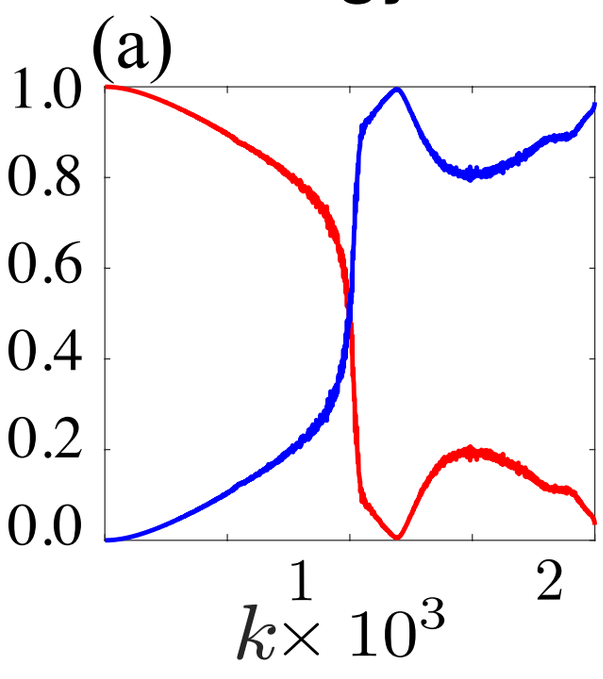
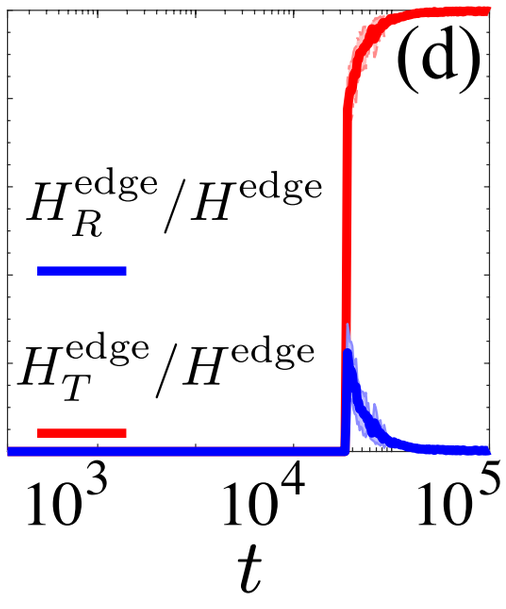
Polarisation



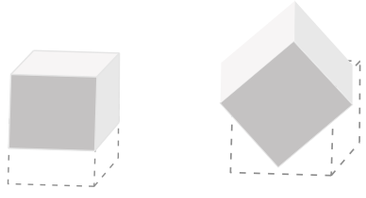


How energy is distributed in the center and the edges





Independent of the initial excitation



Rotation

Transverse disp.

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Outline

- Motivation and Introduction

- A disordered granular lattice vs the FPU model
  - Delocalisation due to nonlinearity
  - Chaos and equipartition

- 1D disorder lattices with 2-DOF (the effect of rotations)
  - Slower spreading due to rotations
  - Polarised spreading

- **Nonlinear flexible metamaterials with strong disorder**
  - Road to almost ballistic spreading

- Perspectives

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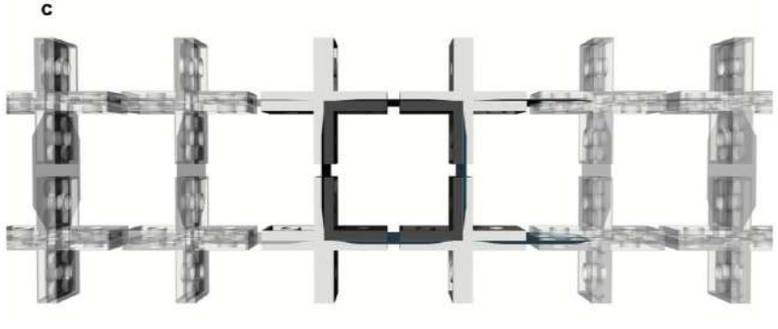
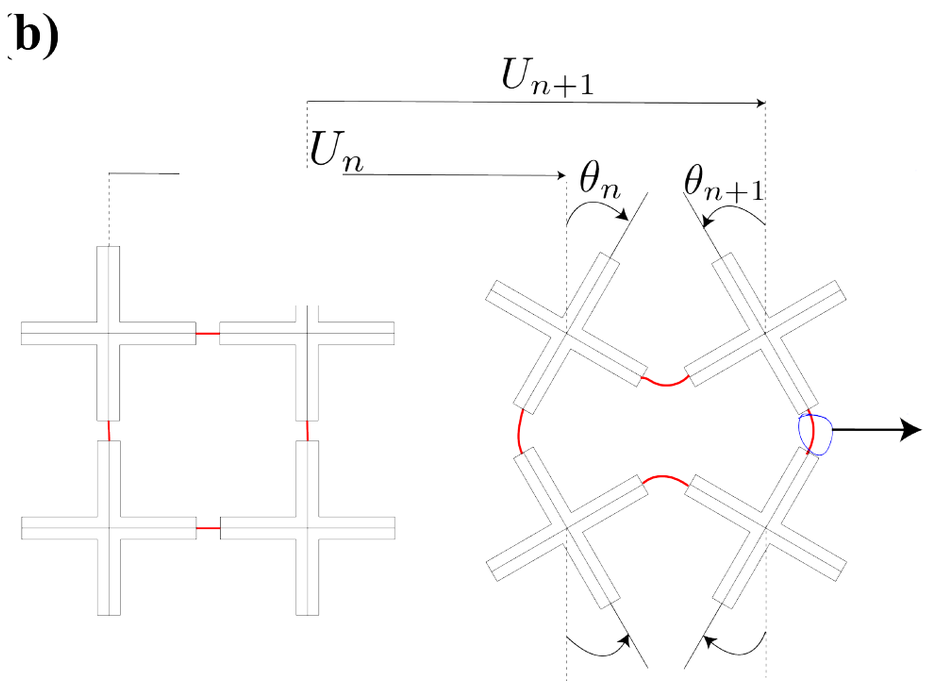
Flexible metamaterial

LEGO lattice in LAUM

Deng *et. al*, *Nat. Comm.* **9** (2018)

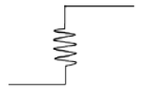


2 dof - 3 different stifness



Bending

Shear



Stretching

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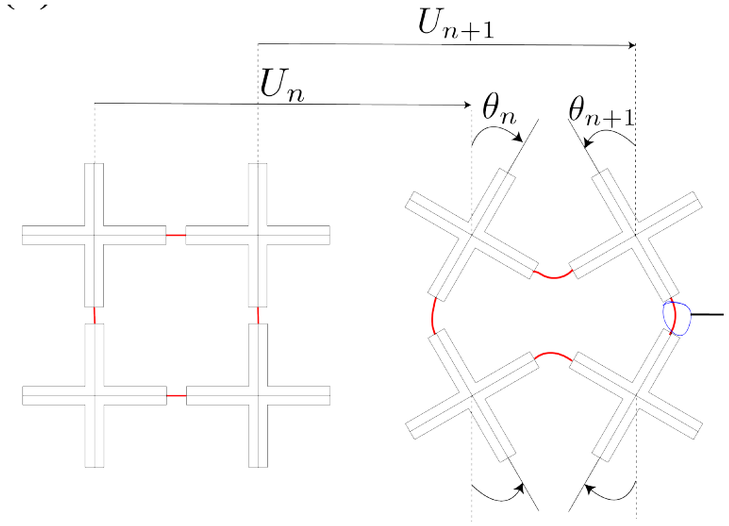
Flexible metamaterial

Strongly nonlinear

$$M_n\ddot{U}_n = \left[U_{n+1} - U_n + \frac{1}{2}\{2 - \cos(\theta_n) - \cos(\theta_{n+1})\}\right] - \left[U_n - U_{n-1} + \frac{1}{2}\{2 - \cos(\theta_n) - \cos(\theta_{n-1})\}\right]$$

$$\begin{aligned}\Gamma_n\ddot{\theta}_n = {} & \frac{1}{4}K^{(s)}\cos(\theta_n)\left[\sin(\theta_{n+1}) - \sin(\theta_n)\right] + \frac{1}{4}K^{(s)}\cos(\theta_n)\left[\sin(\theta_{n-1}) - \sin(\theta_n)\right] \\ & - \frac{1}{4}K^{(\theta)}(\theta_{n+1} + 4\theta_n + \theta_{n-1}) \\ & + \frac{1}{4}\sin(\theta_n)\left[2(U_n - U_{n+1}) + \cos(\theta_n) + \cos(\theta_{n+1}) - 2\right] + \frac{1}{4}\sin(\theta_n)\left[2(U_{n-1} - U_n) + \cos(\theta_n) + \cos(\theta_{n-1}) - 2\right]\end{aligned}$$



Flexible metamaterial

Decoupled in the linear limit

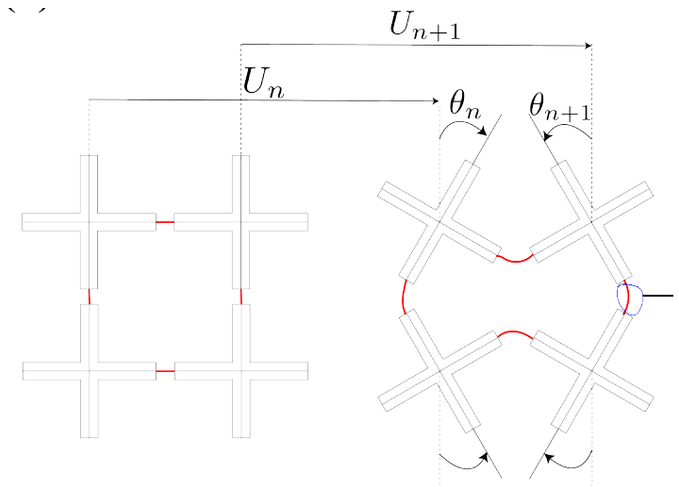
$$M_n \ddot{U}_n = (U_{n+1} - U_n) - (U_n - U_{n-1})$$

$$\Gamma_n \ddot{\theta}_n = \tilde{K}\theta_{n+1} + \tilde{K}\theta_{n-1} - 2\tilde{K}\theta_n - 6K^{(\theta)}\theta_n$$

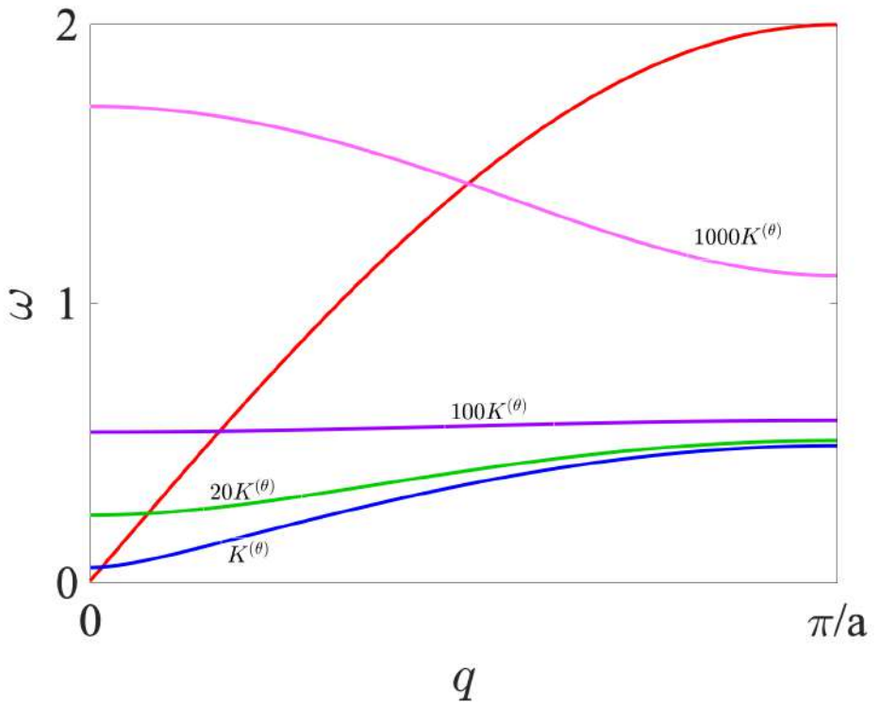
mass spring

+

K.g.



Different values of bending stiffness



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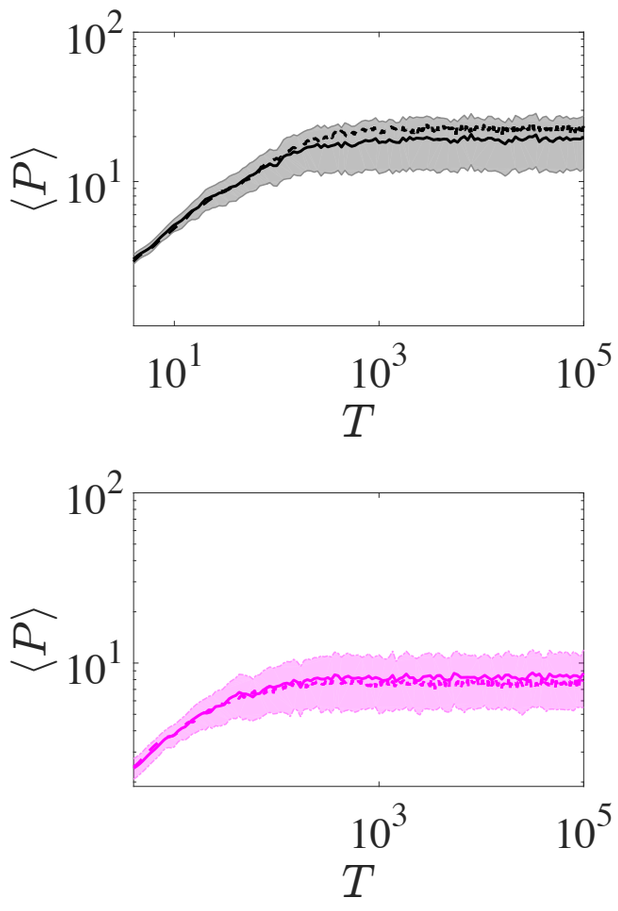
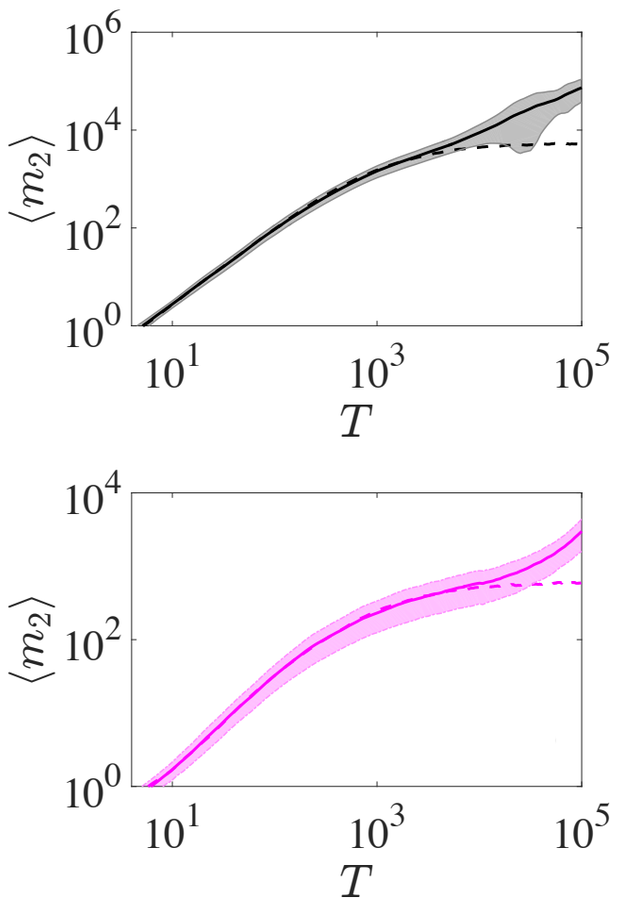
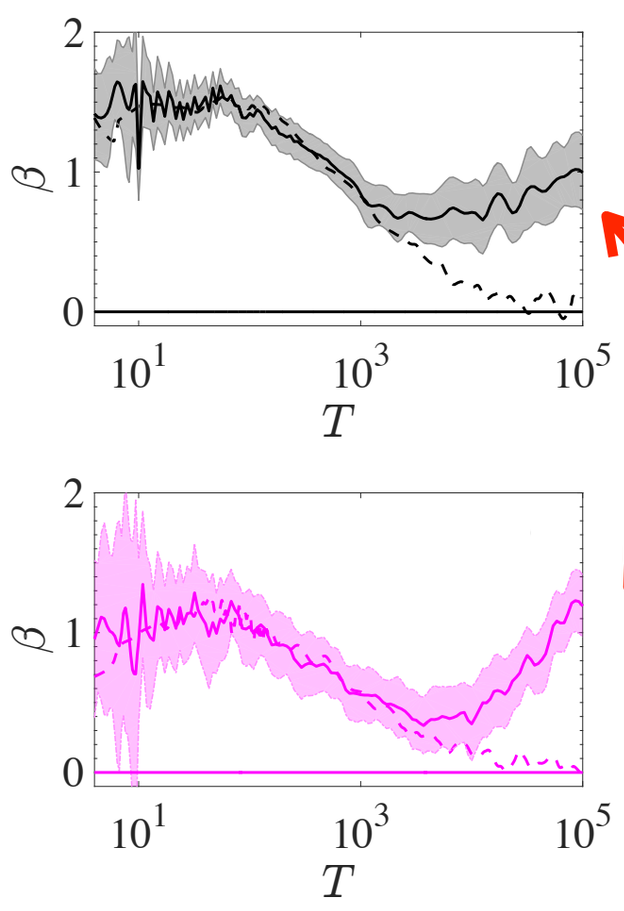
Flexible metamaterial

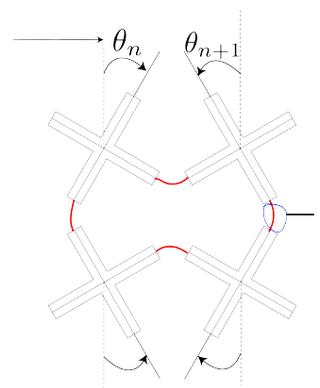
$H = 1.0E\text{-}8$







Not saturate to any value



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Flexible metamaterial

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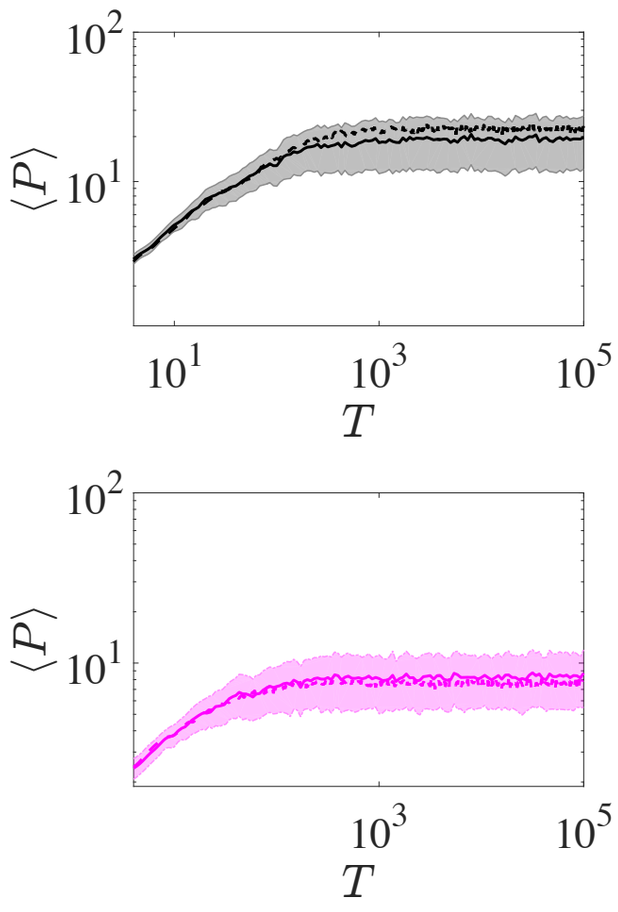
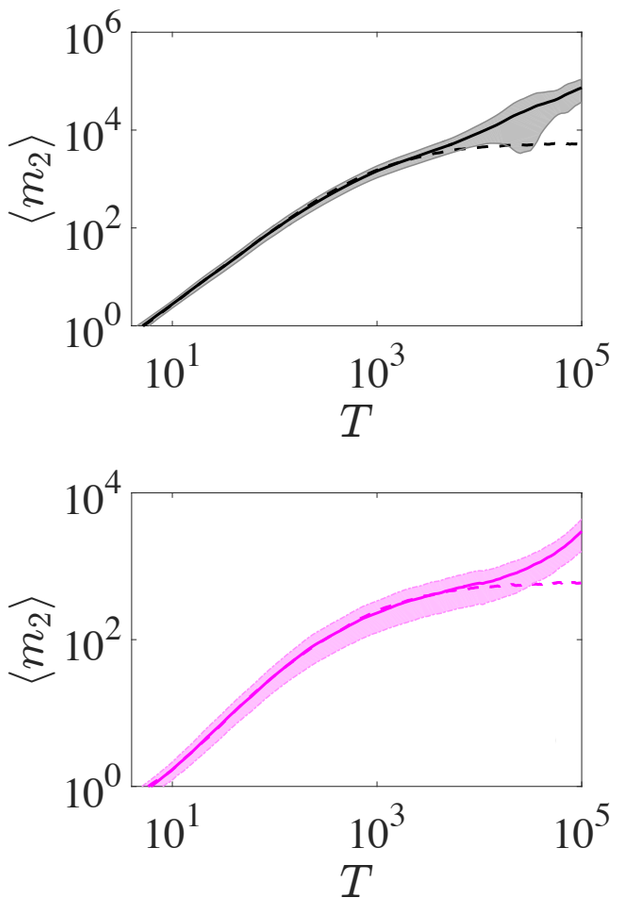
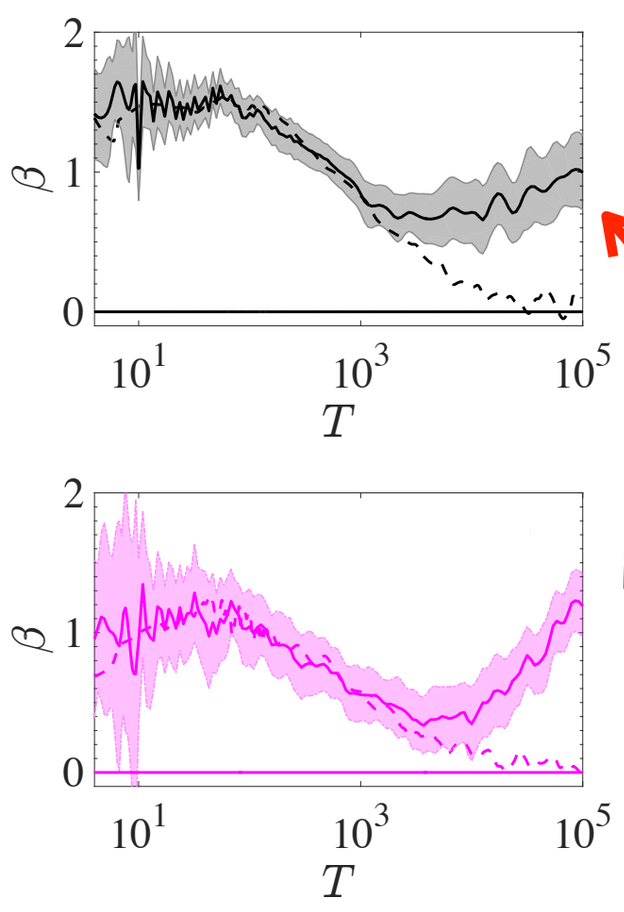
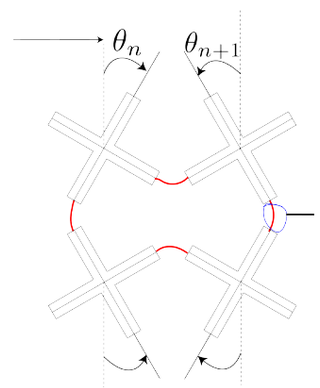






Not saturate to any value



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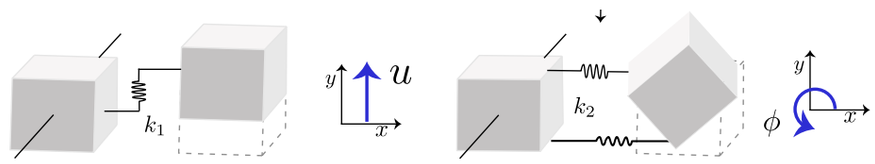
Summary

- Localized chaos
- Equipartition of energy for strong nonlinearities
- Different asymptotic behaviour for the 2 d.o.f. case
- Controlling transport with coupling between the d.o.f.

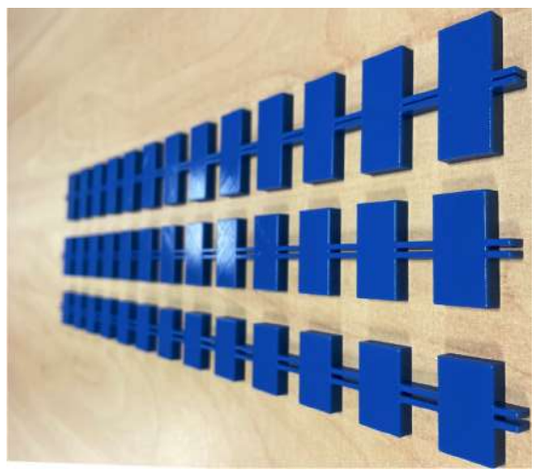
Perspectives

- Theory for the asymptotic behaviour



- Experiments with 2 DOF

Florian Allein



- Other models with various spreading behaviors

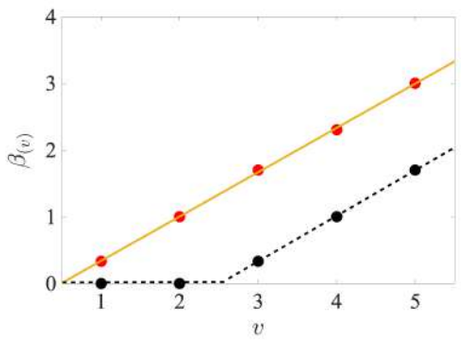
Summary

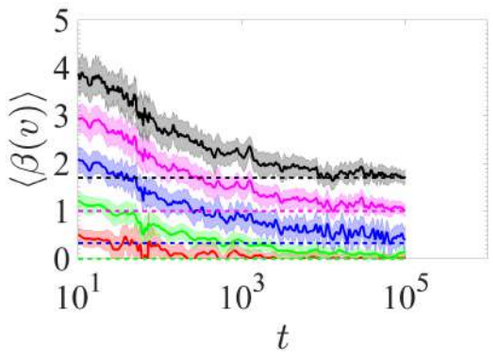
- Localized chaos
- Equipartition of energy for strong nonlinearities
- Different asymptotic behaviour for the 2 d.o.f. case
- Controlling transport with coupling between the d.o.f.

Perspectives

Discrete 4-th order derivative

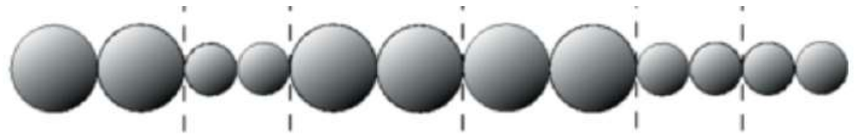
$$m_n \ddot{u}_n = -u_{n-2} + 4u_{n-1} - 6u_n + 4u_{n-1} - u_{n-2}$$





correlated disorder

$$\mathrm{i}\,\dot{\psi}_n = \psi_{n+1} + \psi_{n-1} + V_n \psi_n$$



$$M_q(t) \sim \begin{cases} t^{(2q-1)/3} & (q > 1/2) \\ \ln t & (q = 1/2) \\ \text{finite} & (q < 1/2) \end{cases}$$

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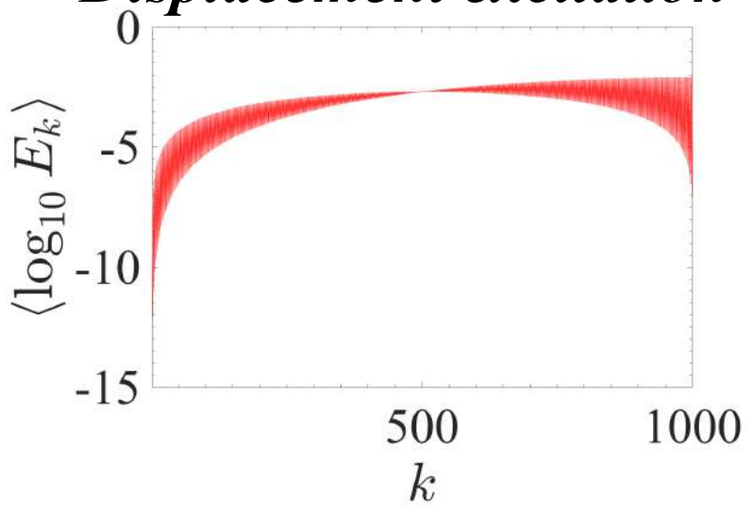
$$\mathbf{R} = \mathbf{A}^{-1}\mathbf{U}_0, \qquad \mathbf{P} = \mathbf{A}^{-1}\mathbf{V}_0, \qquad \text{where,}$$

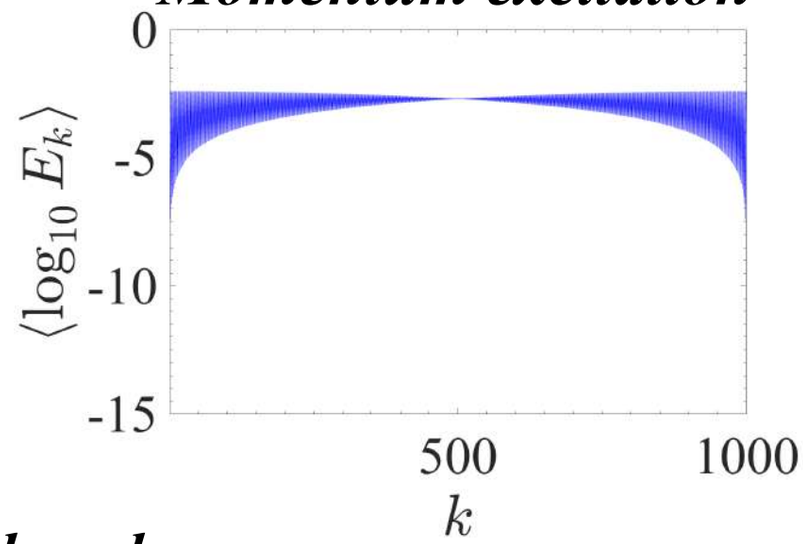
$$\mathbf{U}_0 = [0, \ldots, U_0, \ldots, 0]^T \qquad \mathbf{V}_0 = [0, \ldots, P_0, \ldots, 0]^T,$$

$$E_k = \omega_k^2 R_k^2/2 \qquad E_k = P_k^2/2$$

Displacement excitation

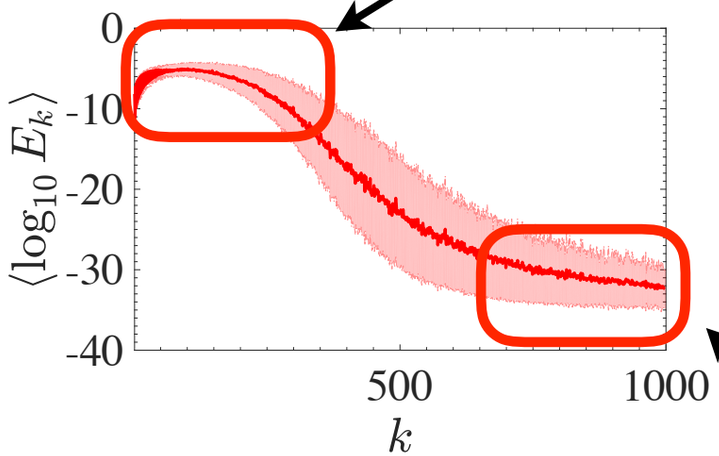


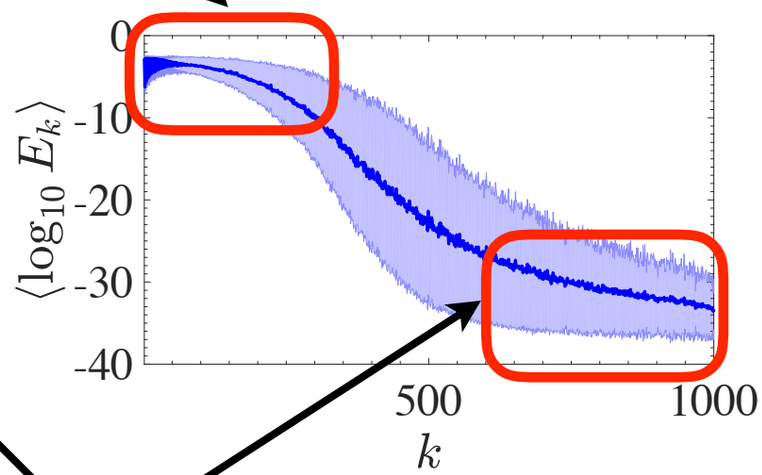
Momentum excitation



Homogeneous: mode energy depends on excitation type

Extended modes





Localized modes

Disordered: Localized modes are less excited

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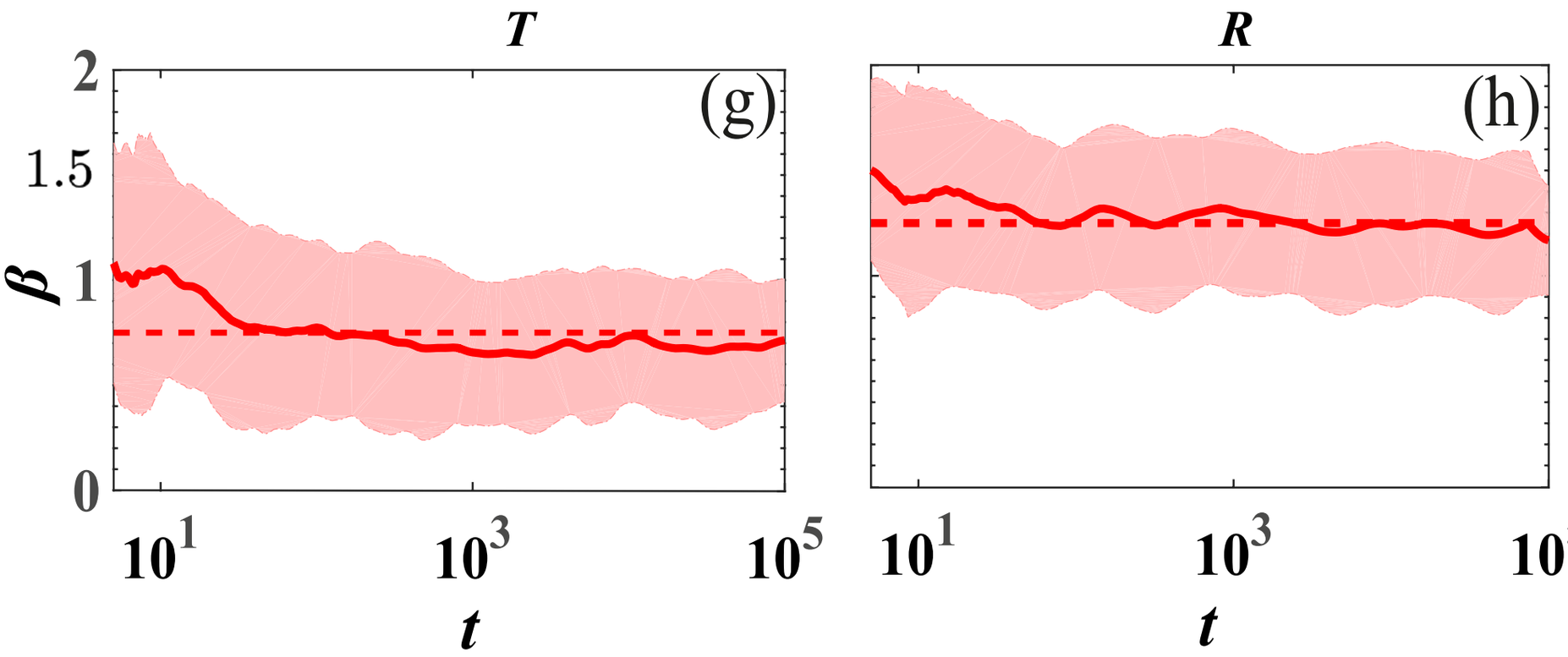
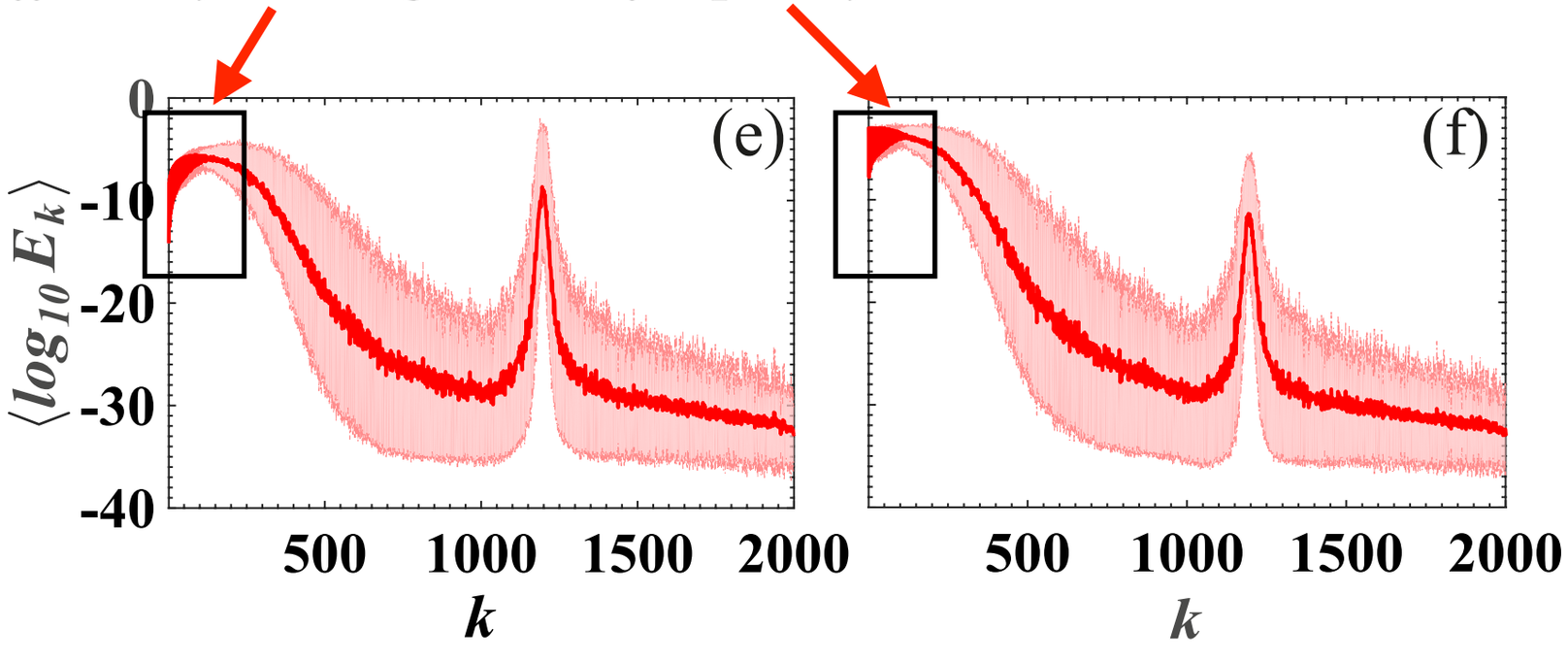
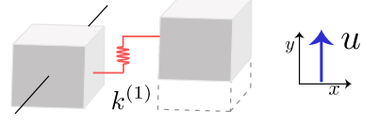
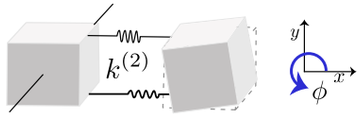
Initial momentum



Differently exciting the low frequency extended modes







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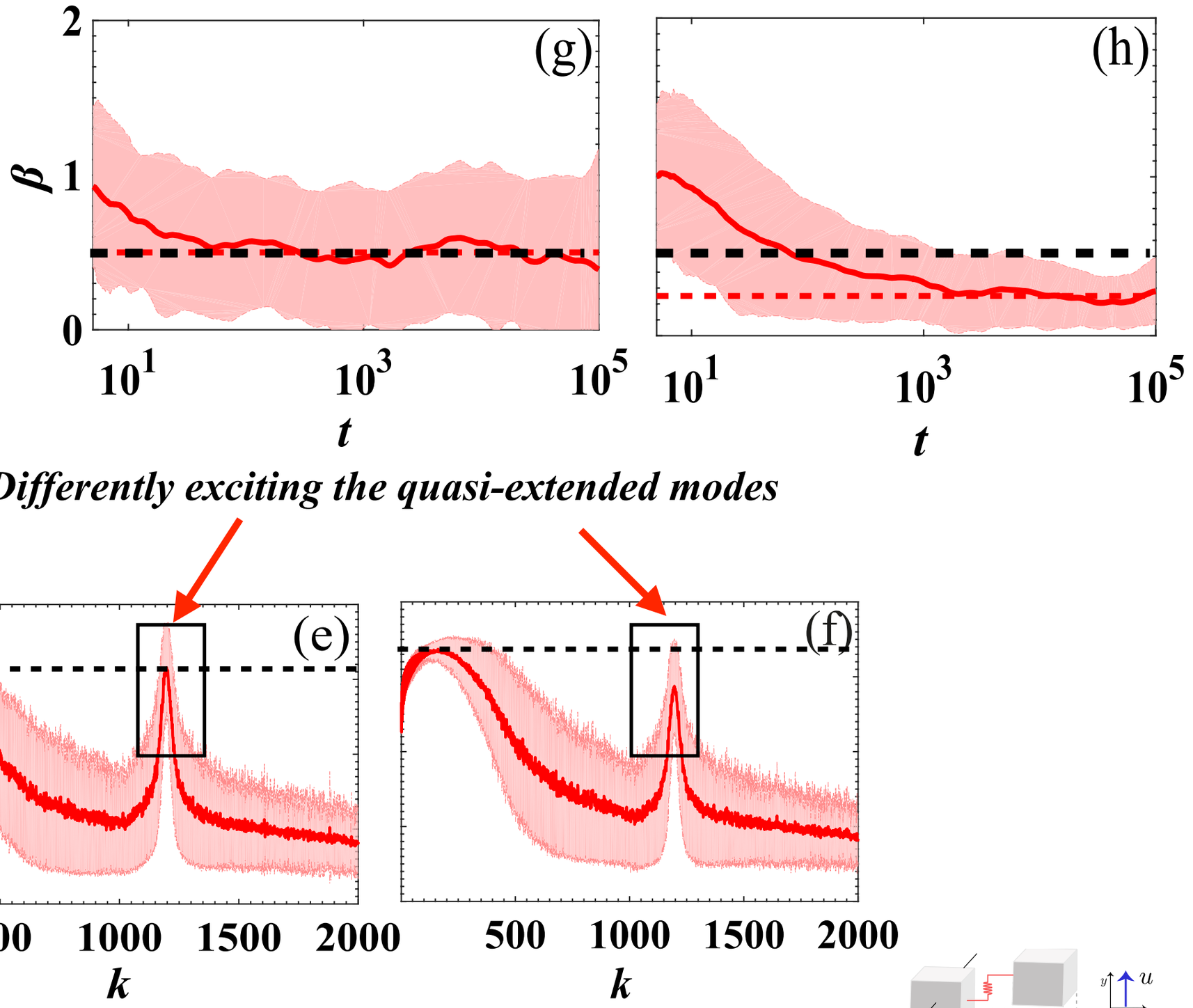
Initial displacement



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1D disordered linear

$$H = \sum_{n=-N/2+1}^{N/2} \frac{{P_n^{(u)}}^2}{2} + \frac{{P_n^{(\phi)}}^2}{2I} + \frac{K_{n+1}^{(1)}}{2}\left[(u_{n+1}-u_n) + \frac{3}{2}(\phi_{n+1}+\phi_n)\right]^2 + \frac{K_{n+1}^{(2)}}{2}(\phi_{n+1}-\phi_n)^2$$

$$m\ddot{u}_n = K_{n+1}^{(1)}\left(u_{n+1}-u_n\right) - K_n^{(1)}\left(u_n-u_{n-1}\right) + \frac{hK_{n+1}^{(1)}}{2}\left(\phi_{n+1}+\phi_n\right) - \frac{hK_n^{(1)}}{2}\left(\phi_n+\phi_{n-1}\right)$$

$$I\ddot{\phi}_n = \Big[\frac{hK_n^{(1)}}{2}\left(u_{n-1}-u_n\right) + \frac{hK_{n+1}^{(1)}}{2}\left(u_n-u_{n+1}\right) - \frac{h^2K_{n+1}^{(1)}}{4}\left(\phi_{n+1}+\phi_n\right) - \frac{h^2K_n^{(1)}}{4}\left(\phi_n+\phi_{n-1}\right)$$

$$+ K^{(2)}\left(\phi_{n+1}-\phi_n\right) - K^{(2)}\left(\phi_n-\phi_{n-1}\right)\Big]$$

$$H_R = \sum_{n=-N/2+1}^{N/2} \frac{{P_n^{(\phi)}}^2}{2I} + \frac{K_{n+1}^{(1)}h^2}{8}(\phi_{n+1}+\phi_n)^2 + \frac{K_{n+1}^{(2)}}{2}(\phi_{n+1}-\phi_n)^2 + \frac{K_{n+1}^{(1)}h}{4}(\phi_{n+1}+\phi_n)(u_{n+1}-u_n)$$

$$H_T = \sum_{n=-N/2+1}^{N/2} \frac{{P_n^{(u)}}^2}{2I} + \frac{K_{n+1}^{(1)}}{2}(u_{n+1}-u_n)^2 + \frac{K_{n+1}^{(1)}h}{4}(\phi_{n+1}+\phi_n)(u_{n+1}-u_n),$$