Density of cracks in heterogeneous materials: Qualitative estimates using transmission eigenvalues with artificial background

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Inverse Scattering Theory

Send incident wave $u_i$

Unknown object
Shape? Material?
Damaged?

Catch scattered wave $u_s$

Goal: retrieve information on the medium from $u_s$.

Examples of application:
- non destructive testing
- tomography
- quality control
- medical imaging
- oceanography
Collecting data

- $e_1$ emits plane wave of direction $\theta$ and of wave length $k$.
- $r$ receives scattered field.

This generates a discretization of a function $F: L^2(S^2) \mapsto L^2(S^2)$. For different samples of $k$, the final data $(F_k)^k > 0$. 

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Collecting data

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Recieves scattered field.
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Emits plane wave of direction \( \theta \) and of wave length \( k \).

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$$F : L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2).$$
Collecting data

For different sample of $k$ is obtained the final data: $(F_k)_{k>0}$.

Emits plane wave of direction $\theta$ and of wave length $k$.

Recieves scattered field.

This generates a discretization of a function $F : L^2(S^2) \rightarrow L^2(S^2)$. 
1 Introduction

2 Mathematical modelisation of the data

3 Extraction of relevant values from the far field operator

4 Application to crack monitoring

5 Conclusion and perspectives
1 Introduction

2 Mathematical modelisation of the data
   • The direct problem
   • The far field operator

3 Extraction of relevant values from the far field operator

4 Application to crack monitoring

5 Conclusion and perspectives
The direct problem

**Notations:**
- \( k \): wave length
- \( u_i \): incident wave
- \( u_s \): scattered wave
- \( u \): total wave

\[
\Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^3 \setminus \Gamma \\
\partial_\nu u = 0 \text{ on } \Gamma \\
u_s \text{ is an outgoing wave.}
\]
The direct problem

The direct problem

Notations:
- $k$ wave length
- $u_i$ incident wave
- $u_s$ scattered wave
- $u$ total wave

For particular incident waves $u_i(\theta, x) = e^{ikx \cdot \theta}$ with $\theta \in S^2$,
The direct problem

Notations:

- $k$: wave length
- $u_i$: incident wave
- $u_s$: scattered wave
- $u$: total wave

Reference medium:

$n(x) \neq 1$

$n(x) = 1$

$\Gamma$

$D$

$u = u_i + u_s$ such that

$\Delta u + k^2 nu = 0$ in $\mathbb{R}^3 \setminus \Gamma$

$\partial_\nu u = 0$ on $\Gamma$

$u_s$ is an outgoing wave.

For particular incident waves $u_i(\theta, x) = e^{ikx \cdot \theta}$ with $\theta \in S^2$, the far field pattern of $u_s$ is

$$u_s(x) = \frac{e^{ikr}}{4\pi r} u_s^\infty(\theta, \hat{x}) + O\left(\frac{1}{r^2}\right), \text{ as } r \to \infty$$

where $r = |x|$ and $\hat{x} = \frac{x}{|x|}$. 

$\Gamma$
The far field operator

More general incident wave \( u_i(x) = v_g(x) \)

\[
\text{for } g \in L^2(\mathbb{S}^2), \quad v_g(x) = \int_{\mathbb{S}^2} g(\theta) e^{ikx.\theta} d\theta
\]

(Herglotz wave)
The far field operator

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for $g \in L^2(S^2)$,

$$v_g(x) = \int_{S^2} g(\theta) e^{ikx \cdot \theta} d\theta$$

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then

$$u_s(x) = \frac{e^{ikr}}{4\pi r} (Fg)(\hat{x}) + O\left(\frac{1}{r^2}\right).$$
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**Definition: Far field operator**

\[
 F : L^2(S^2) \longrightarrow L^2(S^2) \\
 g \longmapsto \int_{S^2} u_s^\infty(\theta, \cdot) g(\theta) d\theta.
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The far field operator

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for \( g \in L^2(S^2) \), \[
  v_g(x) = \int_{S^2} g(\theta) e^{ikx \cdot \theta} \, d\theta
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Definition: Far field operator

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  F : \quad L^2(S^2) \quad \rightarrow \quad L^2(S^2)
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  g \quad \mapsto \quad \int_{S^2} u^\infty_s(\theta, \cdot) g(\theta) \, d\theta.
\]

How do we estimate the crack density from the data \((F_k)_{k>0}\)?
Introduction

Mathematical modelisation of the data

3 Extraction of relevant values from the far field operator
   - Classical transmission eigenvalues
   - Transmission eigenvalues with artificial background

4 Application to crack monitoring
Classical transmission eigenvalues

The far field operator admits a factorization

\[ F_k = G_k H_k \]

where \( H_k : g \mapsto v_g|_D \) and \( G_k : \overline{R(H)} \ni v \mapsto u_\infty^\infty. \)
Classical transmission eigenvalues

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\( v \in \text{Ker}(G_k) \) iff \( v \) satisfies the interior transmission problem:

Find \( (u,v) \in L^2(D) \times L^2(D) \) such that

\[
\begin{align*}
\Delta v + k^2 v &= 0 \quad \text{in } D, \quad (v \in \overline{R(H)}) \\
\Delta u + k^2 n u &= 0 \quad \text{in } D \setminus \Gamma, \\
\partial_v^\pm u &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

(1)
Classical transmission eigenvalues

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\partial^\pm v &= 0 \quad \text{on } \Gamma,
\end{aligned}
\]

along with the transmission conditions:

\[
\begin{aligned}
u - v &= 0 \quad \text{on } \partial D, \\
\partial_v (u - v) &= 0 \quad \text{on } \partial D. \quad (u_\infty^s(v) = 0)
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Classical transmission eigenvalues

The far field operator admits a factorization

\[ F_k = G_k H_k \]

where \( H_k : g \mapsto v_g|_D \) and \( G_k : \overline{R(H)} \ni \nu \mapsto u_\infty^\nu. \)

\( \nu \in \text{Ker}(G_k) \) iff \( \nu \) satisfies the interior transmission problem:

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\begin{align*}
\Delta \nu + k^2 \nu &= 0 \quad \text{in } D, \\
\nu \in R(H) \\
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**Definition: Transmission eigenvalues (TEs)**

\( k > 0 \) is said to be a transmission eigenvalue if there exist a non trivial solution to (1).
TEs as a spectral signature of the probed medium

To what extent can they be exploited?

Feasibility:
- The TEs can be determined from the measured data \((F_k)_{k>0}\).
  
  [L. Audibert, Qualitative Methods for Heterogeneous Media, PhD 2015]

- TEs carry information on the medium
  
  [F. Cakoni, D. Gintides, H. Haddar, The existence of an infinite discrete set of transmission eigenvalues, 2010]

Difficulties:
- The transmission condition is tricky, classical approach:
  \(w := u - v\) satisfies
  \[(\Delta + k^2)(n - 1)^{-1}(\Delta w + k^2 w) = 0\) on \(D \setminus \Gamma\).

- Quadratic in \(k^2\).
- Spectrum of a self-adjoint operator? \((k^2 \in \mathbb{C} \implies \text{loss of information})\).
- It seems difficult to link \(n\) and \(\Gamma\) to the TEs.

To bypass these drawbacks we suggest to rather consider transmission eigenvalues with artificial background.
Transmission eigenvalues with artificial background

[1. Audibert, L. Chesnel, H. Haddar, Transmission eigenvalues with artificial background for explicit material index identification, 2017]

We introduce an artificial background \( \Omega \) such that \( D \subset \Omega \) and consider:

\[
\tilde{u}^\theta = u^\theta_i + \tilde{u}^\theta_s \quad \text{such that} \\
\Delta \tilde{u}^\theta + k^2 \tilde{u}^\theta = 0 \text{ in } \mathbb{R}^3 \setminus \Omega \\
\tilde{u}^\theta = 0 \text{ on } \partial \Omega \\
\tilde{u}^\theta_s \text{ is an outgoing wave}
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Transmission eigenvalues with artificial background (I)

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\tilde{u}^\theta_s \text{ is an outgoing wave}
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Then we define

\[
F_{k}^{\text{art}} = F_k - \tilde{F}_k.
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$\tilde{F}_k$ is a numerically computed data.
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\[
\begin{aligned}
\Delta w + k^2 w &= 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega \\
\Delta w + k^2 nw &= 0 \quad \text{in } \Omega \\
\partial_v w &= 0 \quad \text{on } \partial \Omega \\
[w] &= 0 \quad \text{on } \partial \Omega \\
[\partial_v w] &= -\varphi \quad \text{on } \partial \Omega
\end{aligned}
\]

(2)

\[
\text{w is an outgoing wave}
\]

where $\varphi = \partial_v \tilde{u}^\theta$. 

Transmission eigenvalues with artificial background (II)

Defining

\[ G_k^{art} : \varphi \mapsto w^\infty \]

where \( w \) is the solution of (2) and

\[ H_k^{art} : g \mapsto \partial_\nu \int_{S^2} g(\theta) \tilde{u}(\theta, x) \, ds(\theta), \]

then

\[ F_k^{art} = G_k^{art} H_k^{art}. \]
Defining

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then

\[ F^\text{art}_k = G^\text{art}_k H^\text{art}_k. \]

\( G^\text{art}_k \) is not injective iff \( k^2 \) is an eigenvalue of

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\Delta w + k^2 nw &= 0 \quad \text{in } \Omega \setminus \Gamma \\
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Transmission eigenvalues with artificial background (II)

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w &= 0 \quad \text{on } \partial \Omega.
\end{align*} \tag{3} \]

The spectrum of (3) is made of real positive isolated eigenvalues

\[ 0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_p \leq \cdots \]
Transmission eigenvalues with artificial background

Computation of the eigenvalues $\tau_j$ from the data $F_k^{\text{art}} (l)$

Theoretical foundation

For $z \in \mathbb{R}^3$, we consider the far field equation:

$$F_k^{\text{art}} g \approx \Phi_z^\infty$$

with $\Phi_z^\infty (\hat{x}) = e^{-ikz \cdot \hat{x}}$. (4)
Computation of the eigenvalues $\tau_j$ from the data $F^\text{art}_k (I)$

**Theoretical foundation**

For $z \in \mathbb{R}^3$, we consider the far field equation:

$$F^\text{art}_k g \approx \Phi^\infty_z$$

with $\Phi^\infty_z (\hat{x}) = e^{-ikz.\hat{x}}$. \hfill (4)

**Theorem (solvability of the far field equation)**

Assuming that $k^2$ is not an eigenvalue of (3) the farfield equation is "solvable" if and only if $z \in D$. 
Computation of the eigenvalues $\tau_j$ from the data $F^\text{art}_k (l)$

Theoretical foundation

For $z \in \mathbb{R}^3$, we consider the far field equation:

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Theorem (solvability of the far field equation)

Assuming that $k^2$ is not an eigenvalue of (3) the farfield equation is "solvable" if and only if $z \in D$.

"Solvable" means:

a) there exist a wave $u_i$ for which the scattered field’s far field is $\Phi^\infty_z$

b) $\forall \varepsilon > 0, \exists g \in L^2 (\mathbb{S}^2)$ such that $\| u_i - v_g \| < \varepsilon$. 
Computation of the eigenvalues $\tau_j$ from the data $F_{k}^{\text{art}}$ (II)

We define the cost function

$$J^{\alpha}(g) = \alpha \| H_{k}^{\text{art}} g \|^2 + \| F_{k}^{\text{art}} g - \Phi_{z}^{\infty} \|^2_{L^2(S^2)}$$

Let $g_{z}^{\alpha} \in L^2(S^2)$ be a minimizing sequence of $J^{\alpha}$.

**Theorem**

Assume that $F_{k}^{\text{art}}$ has dense range, then $k^2$ is an eigenvalue of (3) iff the set of points $z$ for which $\| H_{k}^{\text{art}} g_{z}^{\alpha} \|$ is bounded as $\alpha \to 0$ is nowhere dense in $\Omega$.

Consequently the eigenvalues $\tau_j$ are given by peaks in the curve

$$k \mapsto \int_{\omega \subset \Omega} \| H_{k}^{\text{art}} g_{z}^{\alpha} \| \, dz$$

for small values of $\alpha$. 
1 Introduction

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3 Extraction of relevant values from the far field operator

4 Application to crack monitoring
   - Qualitative estimates
   - Retrieving crack density

5 Conclusion and perspectives
Informative relation between the values $\tau_j$ and the physical properties of the medium

The eigenvalues of (3) satisfies

$$
\tau_{j,n} = \min_{W \subset U_{j}^{\Gamma}} \max_{u \in W \setminus \{0\}} \frac{\|n^{-1}\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}
$$

where $U_{j}^{\Gamma}$ denotes the sets of spaces of $\{u \in H^1(\Omega \setminus \Gamma) \mid u = 0 \text{ on } \partial \Omega\}$.

- $n_1 \leq n_2 \implies \tau_{j,n_1}^{\Gamma,n} \geq \tau_{j,n_2}^{\Gamma,n}$
- If $\Gamma_1 \subset \Gamma_2$ then $H^1(\Omega \setminus \Gamma_1) \subset H^1(\Omega \setminus \Gamma_2)$. Consequently

$$
\Gamma_1 \subset \Gamma_2 \implies \tau_{j,1,n}^{\Gamma} \geq \tau_{j,2,n}^{\Gamma}
$$

(this holds true for $\Gamma_1 = \emptyset$).

We focused on the second point in order to make crack monitoring.
Extrapolation of the previous result

Hypothesis:

"The values \( \tau_j \) depends only on \( |\Gamma| \), furthermore each eigenvalues decreases when \( |\Gamma| \) increases."

Numerical Validation:

Figure 1: \( (\tau_j^{\Gamma})_{j=1..40} \) for fixed \( |\Gamma| \).

Figure 2: \( (\tau_j^{\Gamma})_{j=1..40} \) for different \( |\Gamma| \).

We finally define the relevant quantity "crack density" \( d = |\Gamma| \).
Determination of crack density from the data $(F_k)_{k>0}$

1) Given $n$ we discretize the function $|\Gamma| \mapsto (\tau_j)_j \in \mathbb{R}^p$, it consists in solving the eigenvalue problem (3) for different setting of $\Gamma$.

2) Compute the effective eigenvalues $(\tau_j)_j \in \mathbb{R}^p$ with the method described previously.

3) Minimize the function $|\Gamma| \mapsto \|\tau_j - \hat{\tau}_j\|_{\mathbb{R}^p}$.

**Figure**: Example of discretization of $|\Gamma| \mapsto (\hat{\tau}_j)_j$. 
1. Introduction

2. Mathematical modelisation of the data

3. Extraction of relevant values from the far field operator

4. Application to crack monitoring

5. Conclusion and perspectives
Conclusion

Given the initial data \((F_k)_k > 0\) we define a new data \(F_k^{art} = F_k - \widetilde{F}_k\) where \(\widetilde{F}_k\) is a numerically simulated data.

From the new data \(F_k^{art}\) can be extracted special values \((\tau_j)_j\) that carries information on the probed medium. They are the eigenvalues of the simple problem

\[
\begin{align*}
\Delta w + \tau nw &= 0 \quad \text{in } \Omega \setminus \Gamma \\
\partial_\nu w &= 0 \quad \text{on } \Gamma \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]  \tag{6}

We create a data base \((\hat{\tau}_j)_j\) solution of (6) for known \(|\hat{\Gamma}|\).

Information on \(|\Gamma|\) is retrieved by minimazation of \(||\tau_j - \hat{\tau}_j||\).
Perspectives

- Limit phenomenon for $\tau_j^\Gamma$ when $|\Gamma| \to \infty$.

![Figure: 100 localized cracks, first eigenvalue](image1)

![Figure: 10 cracks, 5th eigenvalue](image2)

- Use results on homogenization. Interest: search only cut-off frequencies, this requires less sampling of $F_k$ with respect to $k$.

- Use machine learning tools to learn the correlation between $(n, \Gamma)$ and $(\tau_j)_{j \geq 0}$.
Thank you

Any Questions?