An asymptotic approach to the elastodynamic homogenization of periodic media

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Background

Metamaterials

Metamaterials are artificial composites with special properties that cannot be found in nature.

Acoustic metamaterial (Liu et al. 2000\textsuperscript{a})

Optical metamaterial (Pendry et al. 2006\textsuperscript{b})


Core developments of Willis’ theory

- 1980s, two polarization fields were introduced for a fictional homogeneous comparison (Wills 1980a,b);
- The elastodynamic homogenization theory of Willis was presented in Willis (1997);
- Asymptotic elastodynamic homogenization methods were proposed for periodic media (Bensoussan et al. 1978, Boutin and Auriault 1993, Craster et al. 2010\(^a\));
- Some extensions of Willis’ theory have been proposed. (Milton and Willis 2007, Amirkhizi and Nemat-Nasser 2008\(^b\), Nemat-Nasser et al. 2011, Nassar H. et al. 2015, etc.).


Preliminaries of homogenization theory

Periodic geometry

Consider a lattice $\mathcal{L}$ of the periodic vector space $\mathcal{E}$. The first Brillouin’s zone $T$ is defined: (same definitions for $\mathcal{E}^*$, $\mathcal{L}^*$ and $T^*$)

$$T = \{x \in \mathcal{E} \mid \|x\| < \|x - y\|, y \in \mathcal{L} - \{0\}\}$$

Floquet-Bloch transform

The Floquet-Bloch transform provides the definition

$$f(x) = \int_{k \in T} \tilde{f}_k(x)e^{ik \cdot x} \, dk$$
Motion equation

Using the FB transform for the constitutive relation and the momentum balance

\[(\nabla + ik) \cdot \{C(x) : [(\nabla + ik) \otimes \tilde{u}_k(x)]\}e^{ik \cdot x} + \tilde{f}_k(x)e^{ik \cdot x} = -w^2 \rho(x) \tilde{u}_k(x)e^{ik \cdot x}\]

Effective field

The space average over the unit cell corresponds to the expected value of the wave amplitude at the local region:

\[\langle f(x) \rangle_{FB} = \frac{1}{|T|} \left( \int_{x \in T} \tilde{f}_k(x) dx \right) e^{ik \cdot x} \equiv \langle \tilde{f} \rangle e^{ik \cdot x}\]

**Weighted average** can be obtained via a random coefficient \(w(x, \alpha)\) with \(\langle w \rangle = 1\) and \(\alpha\) being the portion of each phase in the unit \(T\) (Milton and Willis 2007a):

\[f(x) \equiv \langle w \tilde{f}(x) \rangle e^{ik \cdot x}\]

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Homogenization theory

Localization step

The solution of the motion equation has a coupling relation with the effective strain and vector field (Willis 1997a).

\[ \tilde{u} = \langle \tilde{u} \rangle + A : \langle \tilde{e} \rangle + B \cdot \langle \tilde{v} \rangle \]

One approach to obtaining the two localization tensors is to introduce an eigen-strain field \( \gamma \) (FB wave expression are available).

\[ (\nabla + ik) \cdot \{ C : [(\nabla + ik) \otimes s \tilde{u} - \tilde{\gamma}] \} + \tilde{f} = -w^2 \rho \tilde{u} \]

**Green’s function** Introduce the Green’s function \( g \) in order to make the homogenization motion equation solvable.

\[ (\nabla + ik) \cdot \{ C : [(\nabla + ik) \otimes s g] \} + |T| \delta I = -w^2 \rho g \]

---

Homogenization step

The effective fields are simply defined as a volume average over the unit body $T$. The effective constitutive law is specified by

$$\begin{bmatrix} \Sigma \\ P \end{bmatrix} = \begin{bmatrix} C^e & S^1 \\ S^2 & \rho^e \end{bmatrix}_{k,w} \begin{bmatrix} E - \gamma \\ V \end{bmatrix}$$

where the tensor $S^1$ and $S^2$ are the third-order coupling tensors, which depend on the couple $(k, w)$:

$$C^e = \langle C \rangle + \langle C : [(\nabla y + ik) \otimes s A] \rangle, \quad S^2 = iw \langle \rho A \rangle$$
$$S^1 = \langle C : [(\nabla y + ik) \otimes s B] \rangle, \quad \rho^e = \langle \rho \rangle I + iw \langle \rho B \rangle$$

The effective relation is independent of the prescribed initial condition.
Homogenization conditions

**Virtual work condition** The Hill-Mandel relation is still available in the dynamic case:

\[
\int_T \tilde{f}_k \cdot \tilde{u}_k^* dx = \int_T \tilde{F}_k \cdot \tilde{U}_k^*, \quad k \in T
\]

**Effective field condition** The ”slow wave” wavelength \( \lambda \) is greater than the characteristic length of the unit cell length \( 2l \):

\[
\lambda = \left| \frac{2\pi}{k} \right| \geq 2l \quad \Rightarrow \quad |k| \leq \frac{\pi}{l}
\]

**Energy condition** Effective behavior is intended to describe the macroscopic properties of the composite:

\[
\langle \left[ C : (\nabla \otimes^s \tilde{u}) \right] : (\nabla \otimes^s \tilde{u}^*) \rangle \ll \langle \left[ C : (ik \otimes^s \tilde{u}) \right] : (ik \otimes^s \tilde{u})^* \rangle
\]

which presents a relation of an approximation condition (Nassar H. et al. 2015\(^a\)):

\[
w^2 \lesssim \max \left( \frac{c_m^m}{\rho^m} \right) \frac{\pi^2}{l_m^2}
\]

Motion equations

Two-scale representation

It allows to research the macroscopic behaviour of a periodic medium at the microscopic scale (Bensoussan et al.1978). Let us introduce macro-$x$ and micro-$y$ with

$$y = \varepsilon^{-1} x$$

The local motion equation expression:

$$\left(\nabla + ik\right) \cdot \left\{ C(y) : \left[ (\nabla + ik) \otimes^s \tilde{u}(y) \right] \right\} + \tilde{f} = -w^2 \rho(y) \tilde{u}(y)$$

The strain field expression:

$$\epsilon = \epsilon_x + \frac{1}{\varepsilon} \epsilon_y = \nabla_x \otimes^s u + \frac{1}{\varepsilon} \nabla_y \otimes^s u$$

---

Motion equations

Simplify the each order motion equation with the series expansion
\( \tilde{u}^\varepsilon = \sum_r \varepsilon^r \tilde{u}^r \), with \( r \in N \):

\[
\varepsilon^{-2} : \quad \nabla_y \cdot [C : (\nabla_y \otimes s \tilde{u}^0)] = 0
\]

\[
\varepsilon^{-1} : \quad \nabla_x \cdot [C : (\nabla_y \otimes s \tilde{u}^0)] + \nabla_y \cdot [C : (\nabla_x \otimes s \tilde{u}^0 + \nabla_y \otimes s \tilde{u}^1)] = 0
\]

\[
\varepsilon^0 : \quad \nabla_x \cdot [C : (\nabla_x \otimes s \tilde{u}^0 + \nabla_y \otimes s \tilde{u}^1)] + \nabla_y \cdot [C : (\nabla_x \otimes s \tilde{u}^1 + \nabla_y \otimes s \tilde{u}^2)] + f = -w^2 \rho u_0
\]

......

\[
\varepsilon^{n-1} : \quad \nabla_x \cdot [C : (\nabla_x \otimes s \tilde{u}^{n-1} + \nabla_y \otimes s \tilde{u}^n)] + \nabla_y \cdot [C : (\nabla_x \otimes s \tilde{u}^n + \nabla_y \otimes s \tilde{u}^{n+1})] = -w^2 \rho u_{n-1} \quad n \in N^* \]
Solutions

Comparing the orders of the parameter $\varepsilon$, we get the solution of the first four equations:

\[
\begin{align*}
\mathbf{u}_0 &= \tilde{U}_0 \\
\mathbf{u}_1 &= \tilde{U}_1 + \mathcal{X}_1 \nabla_x \tilde{U}_0 \\
\mathbf{u}_2 &= \tilde{U}_2 + \mathcal{X}_1 \nabla_x \tilde{U}_1 + \mathcal{X}_2 \nabla_x^2 \tilde{U}_0 + \mathcal{H}_2 \tilde{f} \\
\mathbf{u}_3 &= \tilde{U}_3 + \mathcal{X}_1 \nabla_x \tilde{U}_2 + \mathcal{X}_2 \nabla_x^2 \tilde{U}_1 + \mathcal{X}_3 \nabla_x^3 \tilde{U}_0 + \mathcal{H}_3 \nabla_x \tilde{f}
\end{align*}
\]

where the series matrices $\mathcal{H}_i$ are derived from the density difference of composite materials:

\[
e.g. \quad \nabla_y \cdot [\mathbf{C} : \nabla_y \mathcal{H}_2(y)] \tilde{f} = (\rho \langle \rho \rangle^{-1} - I) \tilde{f}
\]
Assumption

As mentioned earlier, the body force and external volume loading have a large impact on the effective impedance expressions. Therefore, in the absence of body force, the effective impedance can be simplified so as to reduce to a regular solution equivalent to the work of Boutin and Auriault (1993\textsuperscript{a}):

\[ \tilde{u} = \sum_{i=0}^{n} \varepsilon^i \mathcal{K}_i \nabla_x (\sum_{i=0}^{n} \varepsilon^i \tilde{U}_i) + O(\varepsilon^{n+1}), \quad \mathcal{K}_0 = I, \quad \nabla_x^0 = I \]

Therefore, the average displacement field takes the form

\[ \langle \tilde{u} \rangle = \sum_{i=0}^{n} \varepsilon^i \tilde{U}_i \]

\[ \text{---} \]

Effective impedance

Hierarchical motion equation

Using the displacement expansion \( u^\varepsilon = \sum_n \varepsilon^n u_n \) with \( n \in N \),

\[ \varepsilon^0 : \quad \tilde{Z}^0 \tilde{U}_0 = \tilde{f} \]
\[ \varepsilon^1 : \quad \tilde{Z}^0 \tilde{U}_1 + \tilde{Z}^1 \tilde{U}_0 = \tilde{Z}^1 \tilde{f} \]
\[ \varepsilon^2 : \quad \tilde{Z}^0 \tilde{U}_2 + \tilde{Z}^1 \tilde{U}_1 + \tilde{Z}^2 \tilde{U}_0 = \tilde{Z}^2 \tilde{f} \]

Ignoring the higher order small items,

\[ Z^2 = (I + \varepsilon\tilde{Z}^1 + \varepsilon^2\tilde{Z}^2)^{-1}(\tilde{Z}^0 + \varepsilon\tilde{Z}^1 + \varepsilon^2\tilde{Z}^2) \]

with

\[ \tilde{Z}^n = iw\langle \rho \mathcal{X}_n \rangle iw - ik\langle C(\mathcal{X}_n + \nabla_y \mathcal{X}_{n+1})\rangle ik \cdot (ik)^n, \quad (n = 0, 1, 2, \text{ with } \mathcal{X}_0 = I) \]
\[ \tilde{Z}^n = ik\langle C : (\nabla_y \mathcal{H}_{n+1} + \mathcal{H}_n)\rangle ik + iw\langle \rho \mathcal{H}_n \rangle, \quad (n = 1, 2, \text{ with } \mathcal{H}_1 = 0) \]
Displacement series expression

Set the lowest order expression like $u^e = u_0 + O(\varepsilon)$, the lowest order motion equation has been defined by:

$$Z^0 = k \cdot \langle C \rangle \cdot k - w^2 \langle \rho \rangle I$$

With the same way, set $u^e = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)$,

$$Z^2 \in \{ \gamma Z^2_{\text{min}}, \gamma Z^2_{\text{max}} \}$$

$$Z^2_{\text{min}} = \tilde{Z}^0 + \varepsilon \tilde{Z}^1 + \varepsilon^2 \tilde{Z}^2 - \hat{Z}^1 - \varepsilon \hat{Z}^2$$

$$Z^2_{\text{max}} = \tilde{Z}^0 + \varepsilon \tilde{Z}^1 + \varepsilon^2 \tilde{Z}^2 - \hat{Z}^1$$

With,

$$\tilde{Z}^i = iw \langle \rho {\hat{X}}_i \rangle iw - ik \langle C : {\hat{X}}_i \rangle ik \quad \text{with} \quad {\hat{X}}_0 = I$$

$$\hat{Z}^i = ik \langle C : \nabla_y {\hat{X}}_i \rangle ik \quad \text{with} \quad {\hat{X}}_0 = I$$

$$\gamma = I + \varepsilon ik \langle C : \nabla_y {\mathcal{H}}_2 \rangle + \varepsilon^2 (ik \langle C : {\mathcal{H}}_2 \rangle ik - iw \langle \rho {\mathcal{H}}_2 \rangle)$$
Dispersion relation

The motion equation for a simple two phase periodic structure:

\[
E_n \left( \frac{u_{n+1} - u_n}{a} \right) - E_{n-1} \left( \frac{u_n - u_{n-1}}{a} \right) + f_n = -w^2 a \rho_n u_n
\]

According for the periodic boundary conditions

\[
\delta \nu^4 / 4 - \nu^2 + \sin^2 (ak) = 0
\]

with

\[
w_0^2 = \frac{\langle E \rangle}{\langle \rho \rangle} = \frac{4E_1E_2}{a(\rho_1 + \rho_2)(E_1 + E_2)}, \quad w_i^2 = \frac{E_i}{a \rho_i} \quad (\text{with } i = 1, 2)
\]

\[
\delta = \frac{16E_1E_2 \rho_1 \rho_2}{(E_1 + E_2)^2(\rho_1 + \rho_2)^2} = \left( \frac{w_0}{w_1} \right)^2 \left( \frac{w_0}{w_2} \right)^2, \quad \nu^2 = \frac{(E_1 + E_2)(\rho_1 + \rho_2)}{4E_1E_2}(aw)^2 = \left( \frac{w}{w_0} \right)^2
\]
The width of the “bandgap” is largely influenced by the structure of the composite:

\[ \nu \in \left[ 0, \sqrt{\frac{2 - 2\sqrt{1 - \delta \sin^2(ak)}}{\delta}} \right] \cup \left[ \sqrt{\frac{2 + 2\sqrt{1 - \delta \sin^2(ak)}}{\delta}}, \frac{2}{\sqrt{\delta}} \right], \quad \forall |k| \leq \frac{\pi}{2a} \]
Rewrite the motion equation in the matrix form.

\[ \{ [\nabla]^T [C] [\nabla] - w^2 [\rho] \} [\tilde{u}] = [\mathcal{K}] [\tilde{u}] = \tilde{f} \]

Combine the continuity and periodic conditions for the displacement \( u \) and stress \( \sigma \), and note that \([P][\tilde{u}] = 0\):

\[ [\mathcal{K} + P] [\tilde{u}] = [\tilde{f}] \]

The dispersion relation is defined by “\( \det\{[\mathcal{K} + P]\} = 0 \)”.

Get the dispersion relation as in the work of Nassar H. et al. (2016\(^a\)):

\[
\cos(2ka) = \frac{(\sqrt{\rho_1 E_1} + \sqrt{\rho_2 E_2})^2}{4\sqrt{\rho_1 E_1 \rho_2 E_2}} \cos((\sqrt{\frac{\rho_1}{E_1}} + \sqrt{\frac{\rho_2}{E_2}})wa) - \frac{(\sqrt{\rho_1 E_1} - \sqrt{\rho_2 E_2})^2}{4\sqrt{\rho_1 E_1 \rho_2 E_2}} \cos((\sqrt{\frac{\rho_1}{E_1}} - \sqrt{\frac{\rho_2}{E_2}})wa)
\]

FEM solution

The weak form of the integral equation:

\[ \int_T \mathbf{C} : \tilde{\varepsilon}(\tilde{\mathbf{u}}) : \tilde{\varepsilon}(\delta \tilde{\mathbf{u}})^* dT - w^2 \int_T \mathbf{\rho} \tilde{\mathbf{u}} \cdot \delta \tilde{\mathbf{u}}^* dT = \int_T \tilde{\mathbf{f}} \cdot \delta \tilde{\mathbf{u}}^* dT \]

Set \( \mathcal{L}_b U_b = 0 \) to represent all the periodic boundary conditions. The global motion equation is defined as:

\[
\begin{pmatrix}
[K] & 0 \\
0 & \mathcal{L}_b
\end{pmatrix}
- w^2
\begin{pmatrix}
[M] & 0 \\
0 & 0
\end{pmatrix}
\begin{bmatrix}
U \\
U_b
\end{bmatrix}
=
\begin{bmatrix}
F \\
0
\end{bmatrix}
\]

The effective impedance has the following dispersion relation:

\[
[K]_{\text{glob}} \mathbf{V} = \lambda [M]_{\text{glob}} \mathbf{V}
\]

where the generalized eigenvalue \( \lambda \) represents the square of the angular frequency \( w^2 \) and \( \mathbf{V} \) stands for the generalized eigenvectors.
Two-layer example

The results of the finite element simulation and the analytical solution are compared.

The Young's modulus \((Pa)\), \(E_1 = 3.0e^8, E_2 = 2.0e^{11}\)

The density \((kg/m^3)\): \(\rho_1 = 1.5e^3, \rho_2 = 3.0e^3\)

The unit characteristic size \(l = 5.0e^{-3}(m)\)
Multi-layer example

Comparison with the results by Nemat-Nasser et Srivastava (2011\textsuperscript{a}).

\[ \delta x = 0.0001m \]

\[ \delta x = 0.00005m \]

\[ \delta x = 0.00002m \]

\[ \delta x = 0.00001m \]

Conclusions

- The influence of body force term on the asymptotic effective impedance has been discussed;
- The FEM results has been compared with the analytical results and analysed the influence of mesh size on the numeric result;
- Two higher order asymptotic expressions of the effective impedances have been derived.

Works to be done

- Verify the validity of the high-order effective impedance expressions.
- Study the dynamic homogenization of the motion equation involving the non-uniformly body force function.