An hyperbolic model of nonlinear acoustics with Helmholtz resonators

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1 The model
   - A simplified model

2 Global smooth solutions via Kawashima condition

3 Shock in finite time

4 Global weak entropy solutions
   - splitting scheme
   - entropy solutions
**Solitons**

- **solitons**: nonlinear waves, large amplitude and constant profile
  - ✔ competition between nonlinearity and dispersion
  - ✔ many physical systems: fluid dynamics (KdV), optics, ...

- **acoustics**: no solitons
  - ❌ intrinsic dispersion too low
  - ⚙️ need to introduce geometric dispersion
Acoustic solitons

- tube with Helmholtz resonators
  - Sugimoto (JFM 92, 04) : physical modeling
  - Richoux, Lombard, Mercier (Wave Motion 15) : numerics, experiments

O. Richoux, LAUM
Sugimoto’s model

**Hypotheses**
- low-frequency: propagating plane mode $\rightarrow$ 1D
- weak acoustic nonlinearity in the tube
- $\lambda \gg$ distance $\rightarrow$ continuous distribution of resonators

**Evolution equations**
- field splitted into simple **right-going** (+) and **left-going** waves (-)

\[
\begin{align*}
\frac{\partial u^\pm}{\partial t} + \frac{\partial}{\partial x} \left( \pm au^\pm + b \frac{(u^\pm)^2}{2} \right) &= \pm c \frac{\partial^{-1/2} u^\pm}{\partial t^{1/2}} + d \frac{\partial^2 u^\pm}{\partial x^2} + e \frac{\partial p^\pm}{\partial t} \\
\frac{\partial^2 p^\pm}{\partial t^2} + f \frac{\partial^{3/2} p^\pm}{\partial t^{3/2}} + gp^\pm - m \frac{\partial^2 (p^\pm)^2}{\partial t^2} + n \left| \frac{\partial p^\pm}{\partial t} \right| \left| \frac{\partial p^\pm}{\partial t} \right| &= \pm hu^\pm
\end{align*}
\]

- propagation: linear $a$ and **nonlinear** $b$
- oscillations: linear $g$ and **nonlinear** $m$
- coupling: $e$, $h$
- attenuation: $d$, fractional $c$ and $f$, **nonlinear** $n$
A simplified model

Hypothesis: right-going waves, no fractional attenuation, no nonlinear losses,

\[
\begin{align*}
\partial_t u + a \partial_x u + b \partial_x \left( \frac{u^2}{2} \right) &= -\Omega^2 \varphi, \\
\partial_t \varphi &= u - \varepsilon \varphi - \omega_0^2 \phi, \\
\partial_t \phi &= \varphi,
\end{align*}
\]

\(\Omega > 0, \omega_0 > 0, a > 0, b > 0, \varepsilon \geq 0\), with initial data,

\[u(0, x) = u_0(x), \quad \varphi(0, x) = \varphi_0(x), \quad \phi(0, x) = \phi_0(x),\]

Non increasing energy for smooth solution

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left( u^2 + \Omega^2 \varphi^2 + \Omega^2 \omega_0^2 \phi^2 \right) (x, t) \, dx = -\Omega^2 \varepsilon \int \varphi^2 \, dx.
\]
Rewritten system as an equation

Burgers-Helmholtz system

\[
\partial_t u + a \partial_x u + b \partial_x \left( \frac{u^2}{2} \right) = -\Omega^2 \varphi, \\
\partial_t^2 \varphi + \varepsilon \partial_t \varphi + \omega_0^2 \varphi = \partial_t u
\]

as a Burgers equation with a nonlocal source term

\[
\partial_t u + a \partial_x u + b \partial_x \left( \frac{u^2}{2} \right) = -u_0(x)K_1(t) + \int_0^t K(t - s)u(s, x)ds.
\]
Basics on the Burgers equation

\[ \frac{\partial}{\partial t} u + a \frac{\partial}{\partial x} u + b \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = 0, \quad u(0, x) = u_0(x) \]

1. Maximum principle: \( \inf u_0(x) \leq u(t, x) \leq \sup u_0(x) \)
2. Global smooth solution if and only if \( u_0 \uparrow \)
3. Apparition of a shock wave in a finite time
4. Unique global weak entropy solution \( \forall u_0 \in L^\infty : \) Kruzkov 1970
5. Smoothing effect \( \forall t > 0, \ u(t, .) \in BV_x(\mathbb{R}, \mathbb{R}) : \) Lax & Oleinik 1957
Existence of some small smooth solutions

The system is *partially* dissipative.

The **Kawashima condition (K)** (1985) ensures the existence of global (small) smooth solutions near equilibrium.

(K) means for the linearized system $\frac{\partial}{\partial t} U + A \frac{\partial}{\partial x} U = SU$ at some equilibrium that there is no eigenvectors of the linearized flux in the kernel of the linearized source.

\[
AV = \lambda V \quad \& \quad V \neq \vec{0} \quad \Rightarrow \quad S \ V \neq \vec{0}
\]

If (K) is not satisfied some travelling waves are not dissipated

If $AV = \lambda V$ and $SV = 0$ then the travelling wave $U(t, x) = \nu(x - \lambda t) V$ where $\nu$ is a scalar function is not dissipated by the source term.
Global smooth solutions around non zero equilibrium

Proposition ((K)=(Kawashima) condition)

The (K) condition is fulfilled at all non zero equilibrium which is parametrized by \( \phi_e \neq 0 \),

\[
(u_e, \varphi_e, \phi_e) = (\omega_0^2 \phi_e, 0, \phi_e);
\]

but, the (K) condition is not satisfied at rest

\[
(u_e, \varphi_e, \phi_e) = (0, 0, 0).
\]

and global smooth solutions are expected (?)
Burgers equation with a dissipative source term

With a dissipative source term more smooth solutions than without. The simplest example

\[ \partial_t u + u \partial_x u = -Lu \]

- **global smooth solution** \(\iff -L \leq u'_0(x)\).
- **Lax proof** : \(v = \partial_x u\)

\[(\partial_t + u \partial_x) v = -v^2 - Lv = -v(v + L)\]

\[-\infty \iff -L \rightarrow 0 \iff +\infty\]

\(v = \partial_x u = 0\) attractive equilibrium for all \(-L < u\)

\(\partial_x u = -\infty\) attractive “blow-up” (Lax-shock) for all \(u < -L\)

1. Method generalized by Lax for \(2 \times 2\) system then \ldots Majda \ldots

Theorem (Shock in finite time)

Apparition of shock wave for smooth bounded initial data with

\[ \inf \partial_x u_0(x) \ll 0 \leq \sup \partial_x u_0(x) \ll |\inf \partial_x u_0(x)| \]

A “Lax” proof: \( v = \partial_x u \)

\[
\frac{d v}{d t} + b v^2 = k(t, x) - L(v).
\]

Simplified dynamics along characteristics:

\[
\frac{d v}{d t}(x, t) \leq -b v^2(x, t) + C_0 + C_\varepsilon \sup_{(y, s) \in \mathbb{R} \times [0, t]} |v(y, s)|.
\]
Rewrite an equation as a system

Classic idea: $v(x)$ not smooth, Baiti-Jenssen J.D.E. 1997

$$\partial_t u + \partial_x f(v(x), u) = 0 \iff \begin{cases} 
\partial_t u + \partial_x f(v, u) = 0 \\
\partial_t v = 0
\end{cases}$$

$N \times N$ system: $u = u_1$, $U = (u, u_2, \ldots, u_N)$,

$$\begin{cases} 
\partial_t u + \partial_x f(u) = \sum_{j=1}^{N} a_{1j} u_j, \\
\partial_t u_i = \sum_{j=1}^{N} a_{ij} u_j, \quad 1 < i \leq N
\end{cases}$$
Existence via a splitting scheme

1. Solving on \([t, t + \Delta t]\]

\[
\partial_t u + \partial_x f(u) = 0
\]

2. Next, again on \([t, t + \Delta t]\) with the previous final data at \(t + \Delta t\) as the initial data of \(u = u_1\) at \(t\) for the ODEs

\[
\partial_t U = AU
\]

\[
U = U(t, x).
\]

In general NO “Maximum principle” : No Existence of invariant region
Entropy solutions

$U$ is an entropy solution of the $N \times N$ system if $\forall \eta$ convex, $q' = \eta' f'$,

$$\forall \eta \geq 0, \quad \partial_t \eta(u) + \partial_x q(u) \leq \eta'(u) \sum_{j=1}^{N} a_{1j} u_j \in D'$$

Theorem (Existence and uniqueness of entropy solution)

$\forall T > 0, \exists!$ entropy solution of the $N \times N$ system

$U \in L^\infty([0, T] \times \mathbb{R}, \mathbb{R}^N) \cap C^0([0, T], L^1(\mathbb{R}, \mathbb{R}^N))$

Moreover if the initial data $U_0(x) \in BV^s$, $0 < s \leq 1$, then

$U \in L^\infty([0, T], BV^s(\mathbb{R}, \mathbb{R}^N)) \cap C^s([0, T], L^{1/s}(\mathbb{R}, \mathbb{R}^N))$. 
The same existence and uniqueness result

So the Lax entropy condition on the velocity \([u] \leq 0\)

and no condition on \([\varphi]\) and \([\phi]\)

with a decreasing energy due to shock waves:

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left( u^2 + \Omega^2 \varphi^2 + \Omega^2 \omega_0^2 \phi^2 \right)(x, t) \, dx \leq -\Omega^2 \varepsilon \int \varphi^2 \, dx.
\]

The energy is a "mathematical" entropy

No “Maximum principle”
Conclusions & Prospects

1. A simplified system built on Burgers equation for Burgers-Helmholtz resonators
2. Smooth solutions, Shock Waves, unique entropy solutions
3. More global smooth solutions are expected?
4. Kruzkov 1970: scalar conservation laws, $x = (x_1, \ldots, x_d)$
   \[
   \partial_t u + \text{div} F(t, x, u) = g(t, x, u)
   \]
   Generalisation with variables coefficients?
   for the multi-dimensional case?
5. With fractional time derivatives?
   via the hyperbolic approximated system?