Describing metamaterials with generalized continuum models: the relaxed micromorphic approach

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joint work with:

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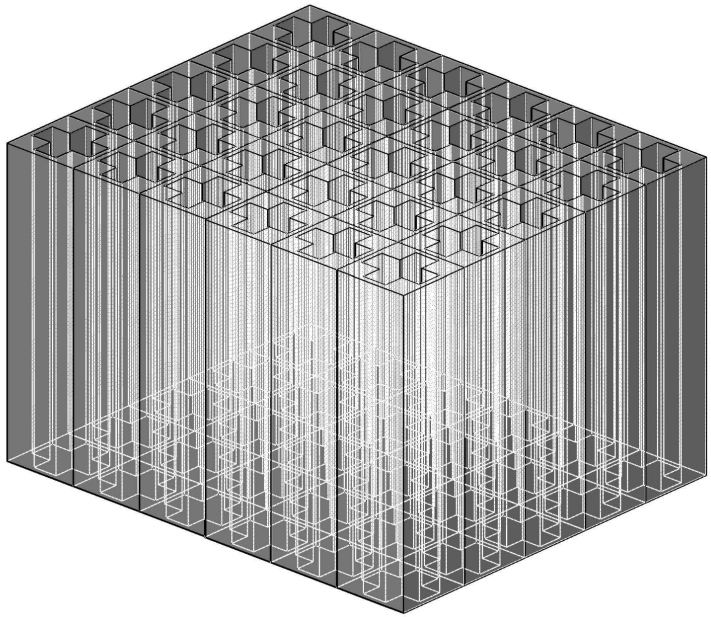
Motivation

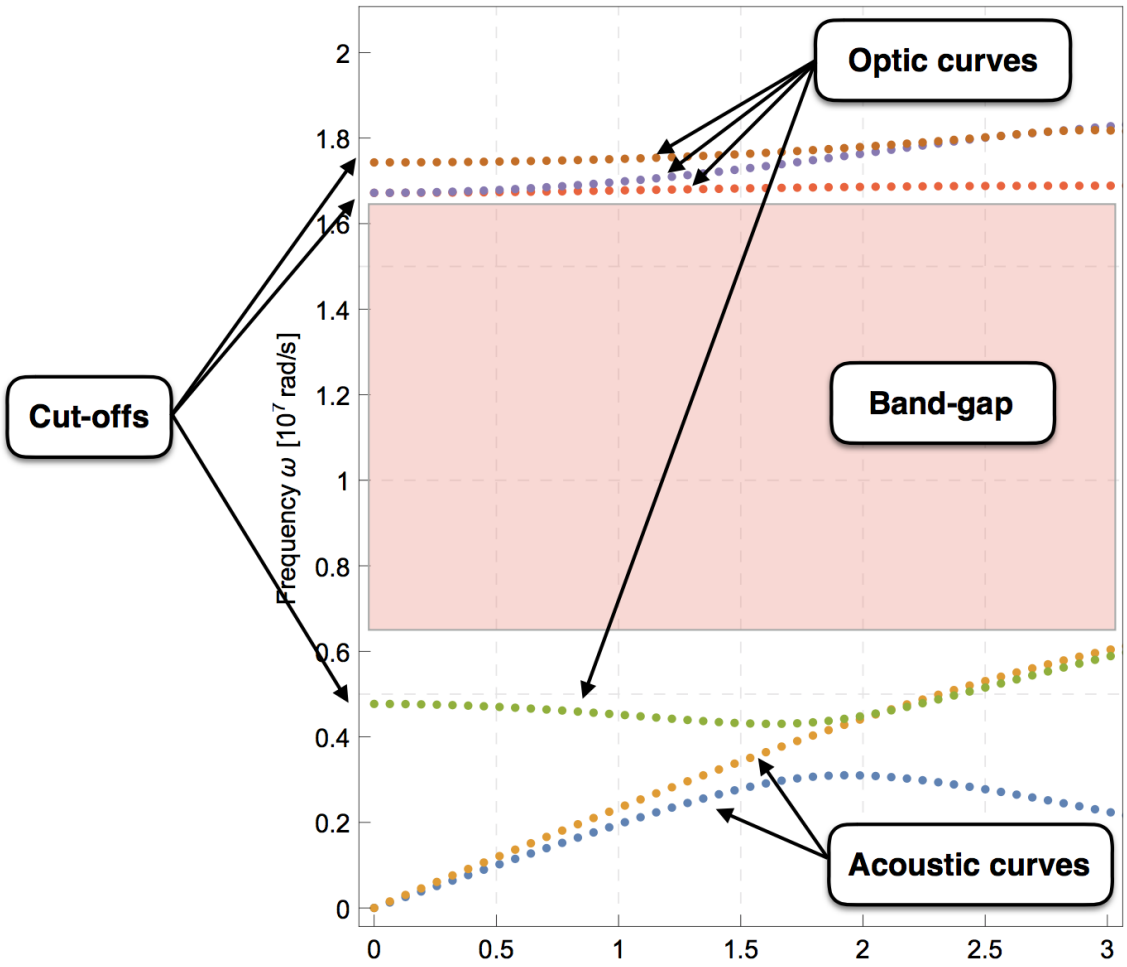
- wave propagation in micro-structured solids,
- non linear dispersion curves, acoustic and optic branches, band-gaps, direction dependent (anisotropic),
- goal: describe "high" frequency response of multi-scale materials without explicitly resolving the micro-structure geometry,
- but taking into account the symmetries of the micro-structure (via **Neumann's principle**).



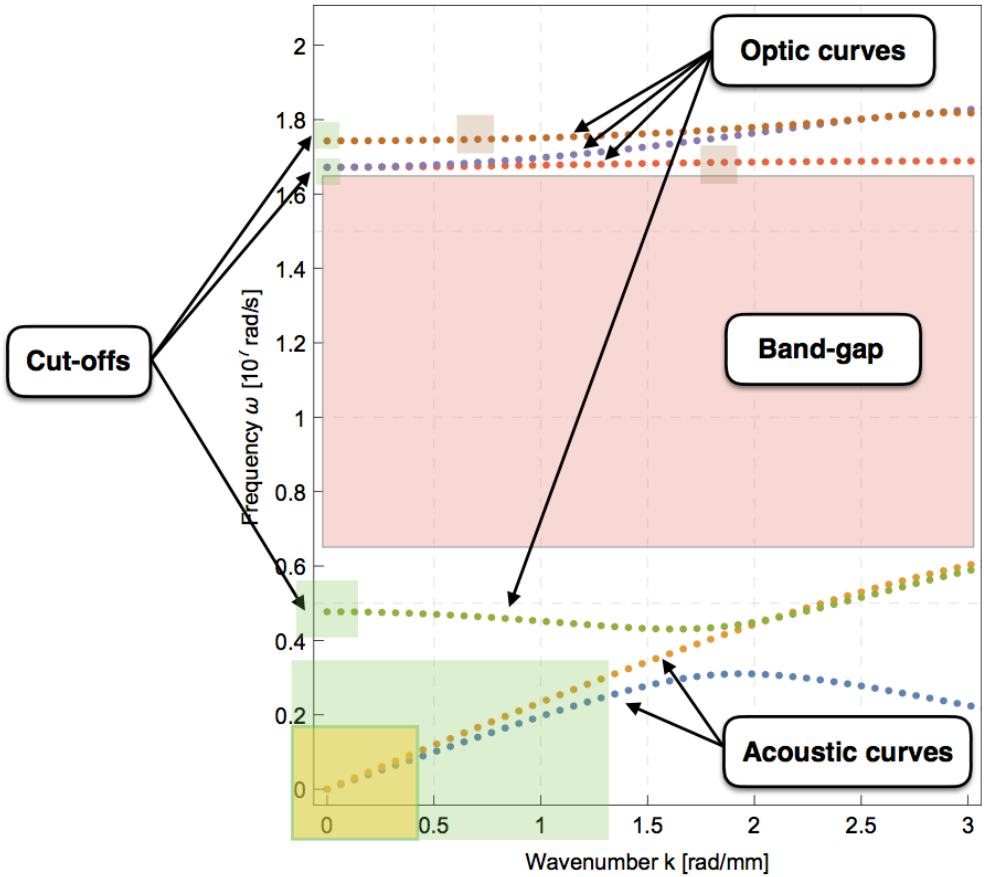
Neumann's principle: how symmetries of the material enter the constitutive law

- Attributed by Voigt (disciple of Neumann),
- **PRINCIPLE: assumed constitutive law describes the physics of the system. Every lattice symmetry of the crystal must also be a symmetry of the assumed constitutive law**.
- In many cases lattice symmetry is equal to symmetry of assumed constitutive law.
- Exception: linear elasticity used to describe hexagonal lattice symmetry, assumed constitutive law will be isotropic. Hexagonal symmetry is subset of isotropy.
- Our assumed anisotropic constitutive law will be based on **Neumann's principle for each constitutive tensor**.

Typical high frequency response - direction dependent



State of the art

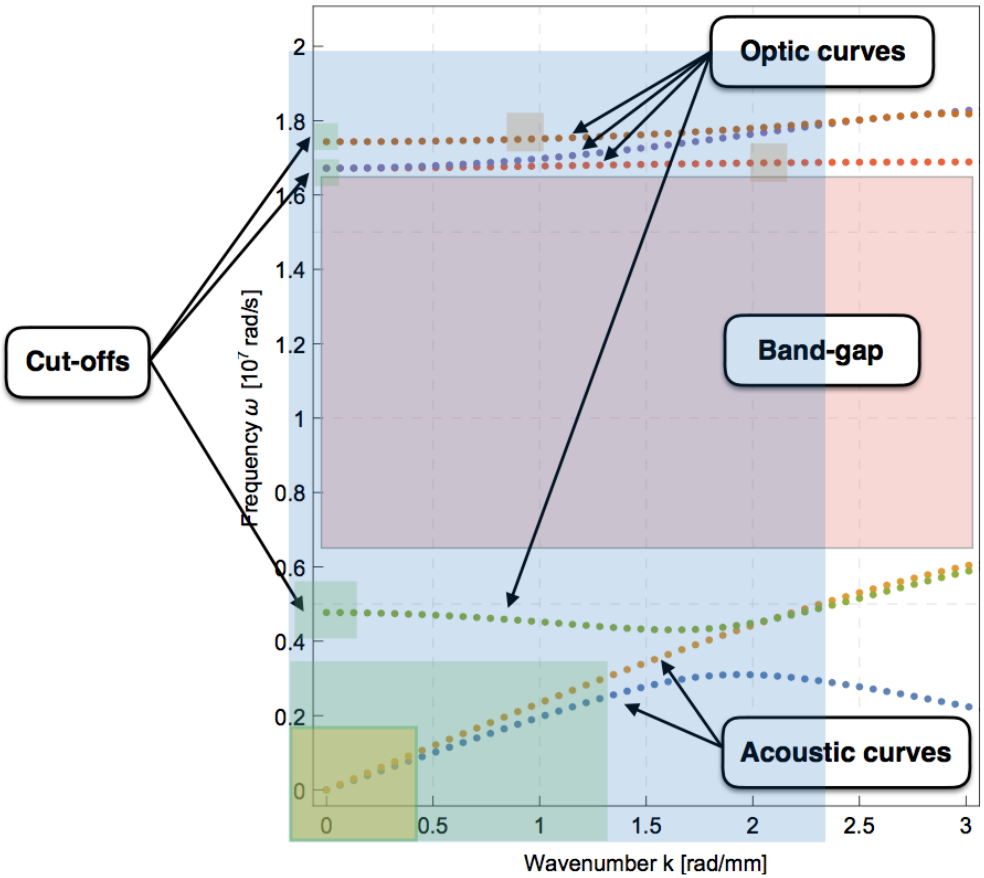


Description by Cauchy elasticity (very long wave length approximation), no dispersion: only straight acoustic curves.

Typical achievement by homogenization methods or gradient elasticity (restricted to acoustic branches).

High frequency homogenization with frequency dependent pseudo-material parameters.

Our goal



relaxed micromorphic model

Outline

- Recapitulation of the **relaxed micromorphic model**.
- Wave propagation in **anisotropic relaxed micromorphic medium**, comparison with dispersion curves for given microstructure based on **Bloch-Floquet analysis**.
- Decisive tool: recently discovered **micro-macro homogenization formula** exclusive to anisotropic relaxed micromorphic medium.
- Decisive tool: **numerical homogenization** on the **unit-cell** level based on **periodic** and **Dirichlet** boundary conditions.
- Reduction of PDE-system to an algebraic system, with **plane-wave ansatz**.
- Comparison of the **dispersion relations** and **polar plots** of phase velocity with results from Bloch-Floquet analysis.

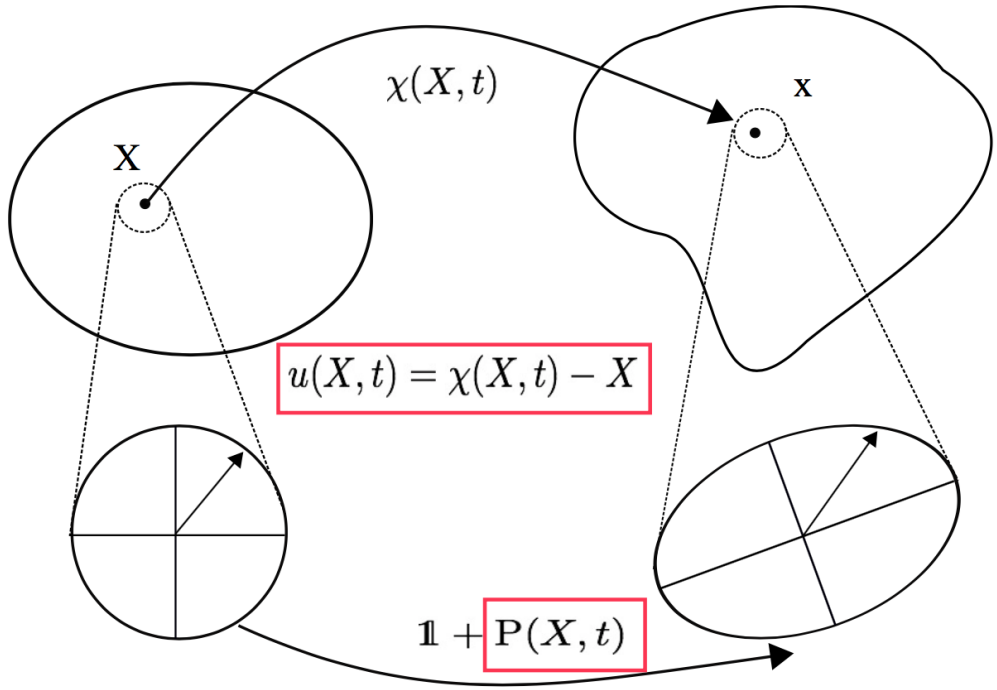
The relaxed micromorphic model

The **kinematics** of the relaxed micromorphic model is described by two fields:

$$u : \Omega \times [0, T] \to \mathbb{R}^3, \qquad P : \Omega \times [0, T] \to \mathbb{R}^{3\times 3},$$

where

- u is the **macroscopic displacement**,
- P is the **micro-distortion tensor field**,
- Ω is the **reference configuration** in $\mathbb{R}^3$.



The classical micromorphic model (Eringen, Mindlin)

The structure of the energy of the classical micromorphic model is

$$\mathcal{E}(\nabla u, P, \nabla P) = \frac{1}{2}\left\langle \widehat{\mathbb{C}}\,(\nabla u - P), \nabla u - P \right\rangle + \frac{1}{2}\left\langle \widetilde{\mathbb{C}}(\nabla u - P), \operatorname{sym} P \right\rangle + \frac{1}{2}\left\langle \mathbb{C}_{\text{micro}}\, \operatorname{sym} P, \operatorname{sym} P \right\rangle + \frac{\mu\, L_c^2}{2}\left\langle \nabla P, \nabla P \right\rangle$$

where

$$\begin{cases} \widehat{\mathbb{C}}, \widetilde{\mathbb{C}} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3} & \text{is dimensionless } 4^{th}\text{order elasticity tensor,} \\ \mathbb{C}_{\text{micro}} : \operatorname{Sym}(3) \to \operatorname{Sym}(3) & \text{is dimensionless } 4^{th}\text{order elasticity tensor,} \\ L_c & \text{characteristic length scale.} \end{cases}$$

Potential energy density

$$W(\nabla u, P, \operatorname{Curl} P) = \underbrace{\frac{1}{2}\left\langle \mathbb{C}_e \operatorname{sym}(\nabla u - P), \operatorname{sym}(\nabla u - P)\right\rangle}_{\text{anisotropic elastic - energy}} + \underbrace{\frac{1}{2}\left\langle \mathbb{C}_{\text{micro}} \operatorname{sym} P, \operatorname{sym} P\right\rangle}_{\text{micro - self - energy}}$$
$$+ \underbrace{\frac{1}{2}\left\langle \mathbb{C}_c \operatorname{skew}(\nabla u - P), \operatorname{skew}(\nabla u - P)\right\rangle}_{\substack{\text{invariant local anisotropic rotational elastic coupling} \\ \text{(related to Cosserat couple modulus } \mu_c)}}$$
$$+ \underbrace{\frac{\mu L_c^2}{2}\left\langle \operatorname{Curl} P, \operatorname{Curl} P\right\rangle}_{\text{curvature}}$$

where

$$\begin{cases} \mathbb{C}_e, \mathbb{C}_{\text{micro}}, \mathbb{L} : \operatorname{Sym}(3) \to \operatorname{Sym}(3) \\ \mathbb{C}_c : \mathfrak{so}(3) \to \mathfrak{so}(3) \\ L_c \end{cases}$$

classical 4^{th}order elasticity tensors,
4^{th} order coupling tensors (6 independent components),
characteristic length scale.

Comparison between the two energies

Eringen-Mindlin energy		Relaxed micromorphic energy
$\mathcal{E}(\nabla u, P, \nabla P)$		$W(\nabla u, P, \operatorname{Curl} P)$
$= \frac{1}{2}\langle \widehat{\mathbb{C}}\,(\nabla u - P), \nabla u - P\rangle$	"orthogonal split"	$= \frac{1}{2}\left\langle \mathbb{C}_e \operatorname{sym}(\nabla u - P), \operatorname{sym}(\nabla u - P)\right\rangle$
		$+ \frac{\mu_c}{2}\left\langle \operatorname{skew}(\nabla u - P), \operatorname{skew}(\nabla u - P)\right\rangle$
$+ \frac{1}{2}\left\langle \widetilde{\mathbb{C}}(\nabla u - P), \operatorname{sym} P\right\rangle$	"no mixed terms"	
$+ \frac{1}{2}\langle \mathbb{C}_{\text{micro}} \operatorname{sym} P, \operatorname{sym} P\rangle$		$+ \frac{1}{2}\langle \mathbb{C}_{\text{micro}} \operatorname{sym} P, \operatorname{sym} P\rangle$
$+ \frac{\mu L_c^2}{2}\langle \nabla P, \nabla P\rangle$	"relaxed"	$+ \frac{\mu L_c^2}{2}\langle \operatorname{Curl} P, \operatorname{Curl} P\rangle$

μ_c : Cosserat couple modulus, can be zero!

$\rightsquigarrow$ Then Cauchy stress is symmetric

Kinetic energy density

$$J(u_{,t}, \nabla u_{,t}, P_{,t}) = \frac{1}{2}\langle \rho\, u_{,t}, u_{,t}\rangle + \frac{1}{2}\langle \mathbb{J}_{\text{micro}}\, \operatorname{sym} P_{,t}, \operatorname{sym} P_{,t}\rangle + \frac{1}{2}\langle \mathbb{J}_c\, \operatorname{skew} P_{,t}, \operatorname{skew} P_{,t}\rangle$$
$$+ \frac{1}{2}\langle \mathbb{T}\, \operatorname{sym} \nabla u_{,t}, \operatorname{sym} \nabla u_{,t}\rangle + \frac{1}{2}\langle \mathbb{T}_c\, \operatorname{skew} \nabla u_{,t}, \operatorname{skew} \nabla u_{,t}\rangle$$
$$+ \frac{\mu\, \widetilde{L}_c^2}{2}\langle \operatorname{Curl} P_{,t}, \operatorname{Curl} P_{,t}\rangle$$

where

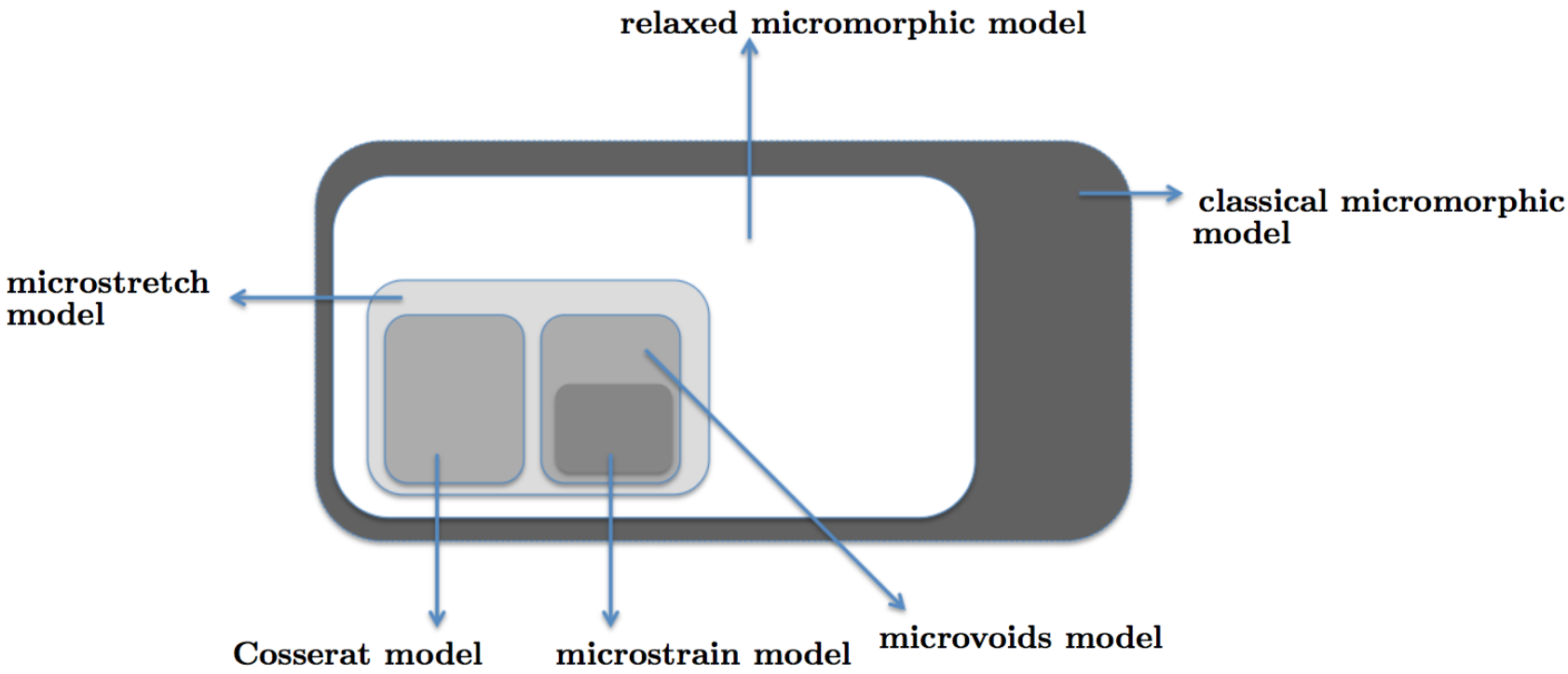
$$\begin{cases} \rho : \Omega \to \mathbb{R}^+ & \text{macro-inertia mass density,} \\ \mathbb{J}_{\text{micro}} : \operatorname{Sym}(3) \to \operatorname{Sym}(3) & \textbf{classical } 4^{th}\text{order tensor (free micro-inertia density),} \\ \mathbb{T} : \operatorname{Sym}(3) \to \operatorname{Sym}(3) & \textbf{classical } 4^{th}\text{order tensor (gradient micro-inertia density),} \\ \mathbb{J}_c, \mathbb{T}_c : \mathfrak{so}(3) \to \mathfrak{so}(3) & 4^{th}\text{order } \textbf{coupling} \text{ tensors (6 independent components),} \\ \widetilde{L}_c & \text{characteristic length scale.} \end{cases}$$

The relaxed micromorphic model

By suitable restriction we obtain **many particular theories of the classical micromorphic model:**

- the linear isotropic microvoids model [Cowin-Nunziato (1983) theory] in dislocation format: $P = \vartheta(x_1, x_2, x_3) \cdot \mathbb{1}$, $\vartheta \in \mathbb{R}$;
- the microstrain model [Forest-Sievert (2006)]: $P \in \mathrm{Sym}\,(3)$;
- the linear isotropic asymmetric microstretch (Eringen's microstretch theory) model in dislocation format;
- the linear isotropic Cosserat model: $P \in \mathfrak{so}\,(3)$;
- the Popov-Kröner (2001) dislocation model;
- the symmetric earthquake structure model of Teisseyre (1973);
- the gauge theory of dislocations of Lazar and Anastassiadis (2008).

The relaxed micromorphic model



- S. Owczarek, I.D. Ghiba, M.V. d'Agostino P. Neff. Non standard micro-inertia terms in the relaxed micromorphic model: well-posedness for dynamics. *arXiv:1809.04791*

First variation and balance equations

The **first variation** of the action functional

$$\delta \int_I \int_\Omega \left(J\left(u_{,t}, \nabla u_{,t}, P_{,t}\right) - W\left(\nabla u, P, \operatorname{Curl} P\right)\right) dv\, dt.$$

yields the **Euler-Lagrange equations** (3 coupled blocks for u, sym P and skew P):

$$\begin{aligned}
&\rho\, u_{,tt} - \operatorname{Div}\left(\mathbb{T}\, \operatorname{sym} \nabla u_{,tt}\right) \\
&\quad -\operatorname{Div}\left(\mathbb{T}_c\, \operatorname{skew} \nabla u_{,tt}\right) = \operatorname{Div}\left(\mathbb{C}_e\, \operatorname{sym}\left(\nabla u - P\right) + \mathbb{C}_c\, \operatorname{skew}\left(\nabla u - P\right)\right), \\
&\qquad \mathbb{J}_{\text{micro}}\, \operatorname{sym} P_{,tt} = \mathbb{C}_e\, \operatorname{sym}\left(\nabla u - P\right) - \mathbb{C}_{\text{micro}}\, \operatorname{sym} P - \mu\, L_c^2\, \operatorname{sym} \operatorname{Curl} \operatorname{Curl} P, \\
&\qquad \mathbb{J}_c\, \operatorname{skew} P_{,tt} = \mathbb{C}_c\, \operatorname{skew}\left(\nabla u - P\right) - \mu\, L_c^2\, \operatorname{skew} \operatorname{Curl} \operatorname{Curl} P.
\end{aligned}$$

whose solutions characterize the equilibrium points of the system.

Why $\operatorname{Curl} P$ in the relaxed micromorphic model?

Simplified statics

$$\int_\Omega \left(\mu_e \left\| \operatorname{sym} (\nabla u - P) \right\|^2 + \mu_c \left\| \operatorname{skew} (\nabla u - P) \right\|^2 + \mu_{\text{micro}} \left\| \operatorname{sym} P \right\|^2 + L_c^2 \left\| \operatorname{Curl} P \right\|^2 \right) dx$$

- **Penalty** $\mu_e, \mu_c \to \infty \Rightarrow P = \nabla u$. Limit energy $\int_\Omega \mu_{\text{micro}} \left\| \operatorname{sym} \nabla u \right\|^2 dx$ $\approx$ **linear elasticity with** $\mathbb{C}_{\text{micro}}$
- **Response of the unit cell:** $L_c \to \infty \Rightarrow \operatorname{Curl} P = 0$, $P = \nabla \vartheta$, $\nabla \vartheta = \nabla u$. Limit energy $\int_\Omega \mu_{\text{micro}} \left\| \operatorname{sym} \nabla u \right\|^2 dx \approx$ **linear elasticity with** $\mathbb{C}_{\text{micro}}$
- **Classical micromorphic** elasticity: penalty $\mu_e, \mu_c \to \infty \Rightarrow$ **second gradient** elasticity $\int_\Omega \left(\mu_{\text{micro}} \left\| \operatorname{sym} \nabla u \right\|^2 + L_c^2 \left\| D^2 u \right\|^2 \right) dx$
- **Classical micromorphic** elasticity: $L_c \to \infty$ **infinite stiffness of the unit cell**: only homogeneous deformations remain possible
- **Classical micromorphic** elasticity: **homogeneous microstructure** $\nabla P = 0 \Rightarrow P = \text{const.}$
- **Relaxed micromorphic** elasticity: **fluctuating microstructure** is possible $\operatorname{Curl} P = 0 \Rightarrow P = \nabla \vartheta$

Well-posedness of the model: static and dynamics

The dynamical problem can be written as a variational problem in a suitable Banach space: considering the spaces of functions $\mathrm{H}_0^1(\Omega)$ and $\mathrm{H}_0(\mathrm{Curl};\Omega)$, where

$$\mathrm{H}_0(\mathrm{Curl};\Omega) := \{P \in \mathrm{L}^2(\Omega,\mathbb{R}^{3\times 3}) \,|\, \mathrm{curl}\, P_i \in \mathrm{L}^2(\Omega,\mathbb{R}^3),\ i=1,2,3 \\ \text{and } P.\tau = 0,\ \forall \tau \text{ tangent to } \partial\Omega\},$$

a weak solution of the given system of PDEs is a function

$$(u,P) \in C^2\Big([0,T]\,;\mathrm{H}_0^1(\Omega)\times \mathrm{H}_0\left(\mathrm{Curl};\Omega\right)\Big)$$

verifying the variational problem

$$\mathcal{W}_1\left((u_{,tt}(t),P_{,tt}(t)),(\varphi,\Phi)\right) = \mathcal{W}_2\left((u(t),P(t)),(\varphi,\Phi)\right) + I^{(f(t),M(t))}(\varphi,\Phi).$$

for all (φ,Φ) and for all $t\in[0,T]$, where $(f,M)\in \mathrm{H}^{-1}(\Omega)\times \mathrm{H}_0(\mathrm{Curl},\Omega)^*$ and

$$I^{(f(t),M(t))}(\varphi,\Phi) := \langle f,\varphi\rangle_{\mathrm{H}^{-1},\mathrm{H}_0^1} + \langle M,\Phi\rangle_{\mathrm{H}_0(\mathrm{Curl})^*,\mathrm{H}_0(\mathrm{Curl})}$$

Well-posedness of the model: static and dynamics

$$\mathcal{W}_1\left((u,P),(\varphi,\Phi)\right) := \int_\Omega \Big(\rho\,\langle u,\varphi\rangle + \langle \mathbb{J}_{\mathrm{micro}}\,\mathrm{sym}\,P, \mathrm{sym}\,\Phi\rangle + \langle \mathbb{J}_c\,\mathrm{skew}\,P, \mathrm{skew}\,\Phi\rangle$$
$$+ \langle \mathbb{T}\,\mathrm{sym}\,\nabla u, \mathrm{sym}\,\nabla\,\varphi\rangle + \langle \mathbb{T}_c\,\mathrm{skew}\,\nabla u, \mathrm{skew}\,\nabla\,\varphi\rangle$$
$$+ \mu\,\widetilde{L}_c^2\,\langle \mathrm{Curl}\,P, \mathrm{Curl}\,\Phi\rangle \Big)\,dx$$

$$\mathcal{W}_2\left((u,P),(\varphi,\Phi)\right) := \int_\Omega \Big(\langle \mathbb{C}_e\,\mathrm{sym}(\nabla u - P), \mathrm{sym}(\nabla\varphi - \Phi)\rangle$$
$$+ \langle \mathbb{C}_c\,\mathrm{skew}(\nabla u - P), \mathrm{skew}(\nabla\varphi - \Phi)\rangle$$
$$+ \langle \mathbb{C}_{\mathrm{micro}}\,\mathrm{sym}\,P, \mathrm{sym}\,\Phi\rangle + \mu\,L_c^2\,\langle \mathrm{Curl}\,P, \mathrm{Curl}\,\Phi\rangle \Big)\,dx$$

With a fixed-point theorem argument, it can be proven that

Theorem

For every $(f, M) \in C^0\big([0,T]; \mathrm{H}^{-1}(\Omega) \times \mathrm{H}_0(Curl;\Omega)^*\big)$, $(u_0, P_0), (\underline{u}_0, \underline{P}_0) \in \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0(Curl;\Omega)$ *and for all tensors* $\mathbb{C}_e, \mathbb{C}_{micro}, \mathbb{C}_c$ *and* $\mathbb{J}_{micro}, \mathbb{J}_c, \mathbb{T}, \mathbb{T}_c$ *which satisfy the given conditions, there exists a unique solution of the induced variational problem.*

Korn-type inequality for incompatible tensor fields

Neff et al. (2011-2016) proved that there exists a constant $C(\Omega) > 0$ such that

$$\forall\, P \in \mathrm{H}(\mathrm{Curl}; \Omega, \mathbb{R}^{3\times 3}), \quad P \times n|_{\partial\Omega} = 0$$

we have the two fowwing estimates:

$$\|P\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\mathrm{sym}\, P\|_{L^2(\Omega)} + \|\mathrm{Curl}\, P\|_{L^2(\Omega)} \right),$$

and

$$\|P\|_{L^2(\Omega)} \leq C(\Omega) \left(\|\mathrm{dev\ sym}\, P\|_{L^2(\Omega)} + \|\mathrm{dev\, Curl}\, P\|_{L^2(\Omega)} \right),$$

If $P = \nabla u$, then it turns into a version of Korn's inequality

$$\forall\, u \in \mathrm{H}^1(\Omega, \mathbb{R}^3),\ \nabla u \times n|_{\partial\Omega} = 0 :\ \|\nabla u\|_{L^2(\Omega)} \leq C(\Omega)\ \|\mathrm{sym}\, \nabla u\|_{L^2(\Omega)}\,.$$

- S. Bauer, P. Neff, D. Pauly and G. Starke. Dev-Div-and DevSym-devCurl-inequalities for incompatible square square tensor fields with mixed boundary conditions. *ESAIM Control Optim. Calc. Var.*, 22(1):112-133, 2016.
- P. Neff, D. Pauly, K.J. Witsch. Poincaré meets Korn via Maxwell: Extending Korn's first inequality to incompatible tensor fields. *J. Diff. Equations.* 258(4):1267-1302, 2014.

Plane wave propagation

Monochromatic plane wave ansatz

$$u(x,t) = \widehat{u}\, e^{i(\langle \boldsymbol{k}, x\rangle - \omega t)}, \qquad P = \widehat{P}\, e^{i(\langle \boldsymbol{k}, x\rangle - \omega t)}, \ \boldsymbol{k} = k\,\widehat{\boldsymbol{k}},$$

- $\widehat{u} = (\widehat{u}_1, \widehat{u}_2, \widehat{u}_3)$ is the **polarization vector** in $\mathbb{R}^3$,
- $\widehat{\boldsymbol{k}} = (k_1, k_2, k_3) \in \mathbb{R}^3$, $\left\|\widehat{\boldsymbol{k}}\right\| = 1$ is the **direction** of wave propagation,
- $k = \|\boldsymbol{k}\|$ is the **wave number** and ω is the **frequency**,
- $\widehat{P} = \left(\widehat{P}_{11}, \widehat{P}_{12}, \widehat{P}_{13}, \widehat{P}_{21}, \widehat{P}_{22}, \widehat{P}_{23}, \widehat{P}_{31}, \widehat{P}_{32}, \widehat{P}_{33}\right) \in \mathbb{R}^{3\times 3}$.

The PDE system reduces to:

$$D(k,\omega)\cdot v = 0,$$

where $D(k,\omega)$ is a 12×12 matrix and v is the vector of amplitudes given by the components of $\hat{u}$ and $\hat{P}$. Calculating

$$\det D(k,\omega) = 0,$$

we find the dispersion curves $\omega = \omega(k)$.

Necessary and sufficient conditions for real wave propagation in the isotropic case

Sylvester's criterion states that a Hermitian matrix M is positive-definite if and only if the leading principal minors are positive. Hence, it is possible to prove the following proposition:

The dynamic relaxed micromorphic model admits **real wave velocity** $\frac{\omega}{k}$ *if and only if*

$$\mu_c \geq 0, \qquad \mu_e > 0, \qquad 2\,\mu_e + \lambda_e > 0,$$
$$\mu_{\text{micro}} > 0, \qquad 2\,\mu_{\text{micro}} + \lambda_{\text{micro}} > 0,$$
$$(\mu_{\text{macro}} > 0), \qquad 2\,\mu_{\text{macro}} + \lambda_{\text{macro}} > 0,$$
$$\kappa_e + \kappa_{\text{micro}} > 0, \qquad 4\,\mu_{\text{macro}} + 3\,\kappa_e > 0.$$

The requirement $\mu_{\text{macro}} > 0$ is redundant, since it is implied by $\mu_e, \mu_{\text{micro}} > 0$ and the **Cosserat couple modulus** μ_c only needs to be non-negative for **real wave velocity** $\frac{\omega}{k}$.

Necessary and sufficient conditions for real wave propagation

The necessary and sufficient conditions for real wave propagation are already implied by the smaller set:

$$\mu_e > 0, \qquad \mu_{\text{micro}} > 0, \qquad \mu_c \geq 0, \qquad \kappa_e + \kappa_{\text{micro}} > 0,$$

$$2\,\mu_{\text{macro}} + \lambda_{\text{macro}} > 0.$$

It is easy to notice that:

positive definiteness of the energy $\Longrightarrow$ real wave propagation

- P. Neff, A. Madeo, G. Barbagallo, M. V. d'Agostino, R. Abreu, I.-D. Ghiba, (2017) Real wave propagation in the isotropic-relaxed micromorphic model. **Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences**. DOI: 10.1098/rspa.2016.0790

Legendre-Hadamard (strong) ellipticity conditions

- **Classical elasticity**: $W = W(\nabla u)$

$$\forall\, \xi, \eta \in \mathbb{R}^3 \backslash \{0\}: \quad t \to W(\nabla u + t\, \xi \otimes \eta)$$
$$\text{is convex in } t \text{ at all } \nabla u.$$

Legendre-Hadamard $\Longrightarrow$ **real wave propagation**
strong ellipticity

- **Micromorphic elasticity**: $W = W(\nabla u, P, \nabla P)$

Considering $z = (u, P) \in \mathbb{R}^{12}$, $\nabla z \in \mathbb{R}^{36} \simeq \mathbb{R}^{12} \otimes \mathbb{R}^3$:

$$\forall\, \xi \in \mathbb{R}^3 \backslash \{0\}, \widetilde{\eta} \in \mathbb{R}^{12} \backslash \{0\}: \quad t \to W(z, \nabla z + t\, \xi \otimes \widetilde{\eta})$$
$$\text{is convex in } t \text{ at all } (z, \nabla z).$$

Legendre-Hadamard $\not\Longrightarrow$ **real wave propagation**
strong ellipticity

Strong ellipticity for the relaxed micromorphic model (isotropic case)

The relaxed micromorphic **energy** $W(\nabla u, P, \operatorname{Curl} P)$ is:

$$W = \mu_e \left\| \operatorname{sym} (\nabla u - P) \right\|^2 + \frac{\lambda_e}{2} \operatorname{tr} (\nabla u - P)^2 + \mu_c \left\| \operatorname{skew} (\nabla u - P) \right\|^2$$
$$+ \mu_{\text{micro}} \left\| \operatorname{sym} P \right\|^2 + \frac{\lambda_{\text{micro}}}{2} (\operatorname{tr} (P))^2 + \frac{\mu L_c^2}{2} \left\| \operatorname{Curl} P \right\|^2 .$$

The **strong ellipticity** condition for the relaxed micromorphic model is:

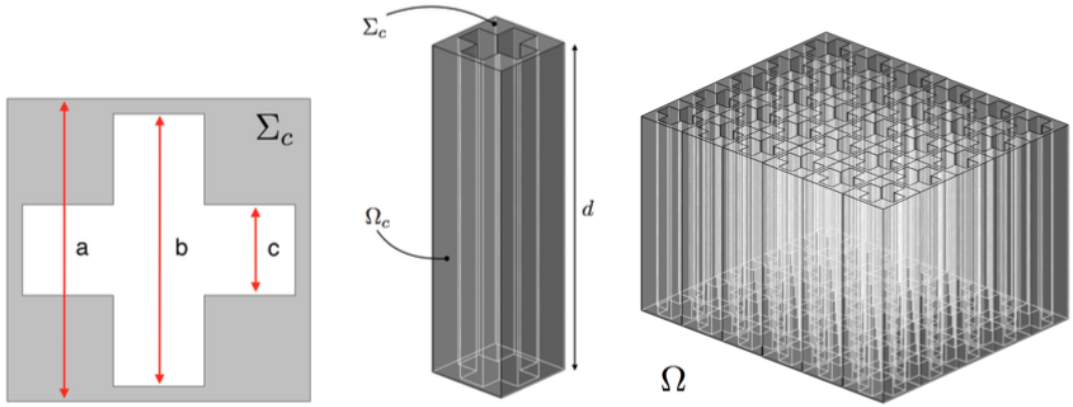
$$\textbf{strong ellipticity} \iff \mu_e + \mu_c > 0 , \quad 2\mu_e + \lambda_e > 0 .$$

This condition is not sufficient to imply real wave propagation:

$$\textbf{strong ellipticity} \not\Longrightarrow \textbf{real wave propagation}$$

Unit-cell and Bloch-Floquet analysis

The **metamaterial** Ω is a periodic medium with **tetragonal** symmetric unit cell.



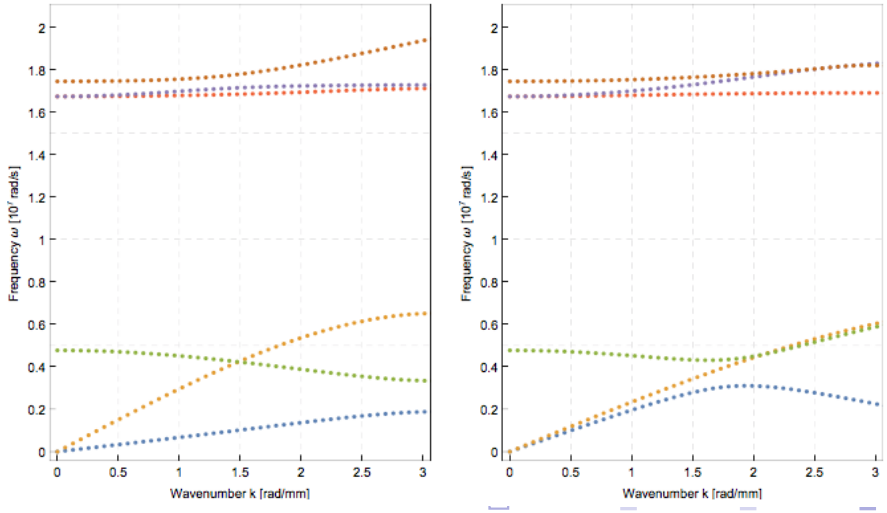
Grey region of Σ_c is filled by aluminum while the white one is empty, ρ is 1485 kg/m^3

a	b	c	d	E	ν
1	0.9	0.3	1	70	0.33
mm	mm	mm	m	GPa	–

Dispersion curves via the Bloch-Floquet analysis for wave propagation directions

$\widehat{\boldsymbol{k}} = (1, 0, 0)$

$\widehat{\boldsymbol{k}} = (\sqrt{2}/2, \sqrt{2}/2, 0)$.



Tetragonal case

The group of the tetragonal symmetry is the **Dihedral group** $\mathbb{D}_4$:

$$Q_{ai}\,Q_{bj}\,Q_{ch}\,Q_{dk}\,\mathbb{C}_{abcd} = \mathbb{C}_{ijhk}, \qquad \forall Q \in \mathbb{D}_4.$$

The matrices corresponding to the considered tensors (Voigt notation) are

$$\widetilde{\mathbb{C}}_e = \begin{pmatrix} 2\mu_e+\lambda_e & \lambda_e & \lambda_e^* & 0 & 0 & 0 \\ \lambda_e & 2\mu_e+\lambda_e & \lambda_e^* & 0 & 0 & 0 \\ \lambda_e^* & \lambda_e^* & (\widetilde{\mathbb{C}}_e)_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & (\widetilde{\mathbb{C}}_e)_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & (\widetilde{\mathbb{C}}_e)_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_e^* \end{pmatrix}, \quad \widetilde{\mathbb{C}}_c = \begin{pmatrix} 4\mu_c^* & 0 & 0 \\ 0 & 4\mu_c^* & 0 \\ 0 & 0 & 4\mu_c \end{pmatrix},$$

(same structure for $\widetilde{\mathbb{C}}_{\text{micro}}, \widetilde{\mathbb{L}}, \widetilde{\mathbb{J}}_{\text{micro}}, \widetilde{\mathbb{T}}$), (same for $\widetilde{\mathbb{L}}_c, \widetilde{\mathbb{J}}_c, \widetilde{\mathbb{T}}_c$).

Tetragonal case

Wave propagation in the plane $(x_1, x_2, 0)$: further reduction to

$$\widetilde{\mathbb{C}}_e = \begin{pmatrix} 2\mu_e + \lambda_e & \lambda_e & * & 0 & 0 & 0 \\ \lambda_e & 2\mu_e + \lambda_e & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_e^* \end{pmatrix}, \quad \widetilde{\mathbb{C}}_c = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 4\mu_c \end{pmatrix},$$

Cosserat couple modulus μ_c

The remaining parameters of the **tetragonal relaxed micromorphic model** are

$$(\mu_e, \lambda_e, \mu_e^*, \mu_c, \mu_{\text{micro}}, \lambda_{\text{micro}}, \mu_{\text{micro}}^*, L_c) \text{ and } (\rho, \eta_1, \eta_2, \eta_3, \eta_1^*, \overline{\eta}_1, \overline{\eta}_2, \overline{\eta}_3, \overline{\eta}_1^*)$$

and we set the characteristic length scales $L_c = \widetilde{L}_c = 0$.

Micro-macro homogenization formula (tensor harmonic mean)

For $L_c \to 0$ the relaxed micromorphic continuum approximates a classical **macroscopic Cauchy medium** whose elasticity tensor $\mathbb{C}_{\text{macro}}$ is related to $\mathbb{C}_{\text{micro}}$ and $\mathbb{C}_e$ of the relaxed micromorphic model via the "Reuss-like" **homogenization formula** (harmonic tensor mean)

$$\mathbb{C}_{\text{macro}} = \mathbb{C}_{\text{micro}} \, (\mathbb{C}_{\text{micro}} + \mathbb{C}_e)^{-1} \, \mathbb{C}_e \quad \Longleftrightarrow$$
$$\mathbb{C}_e = \mathbb{C}_{\text{micro}} \, (\mathbb{C}_{\text{micro}} - \mathbb{C}_{\text{macro}})^{-1} \, \mathbb{C}_{\text{macro}}.$$

Indeed, starting from the system

$$\operatorname{Div}\left[\mathbb{C}_e \operatorname{sym}(\nabla u - P) + \mathbb{C}_c \operatorname{skew}(\nabla u - P)\right] = 0, \quad (1)$$
$$\mathbb{C}_e \operatorname{sym}(\nabla u - P) + \mathbb{C}_c \operatorname{skew}(\nabla u - P) - \mathbb{C}_{\text{micro}} \operatorname{sym} P - \mu \, L_c^2 \, (\operatorname{CurlCurl} P) = 0,$$

in the limit $L_c \to 0^+$, we obtain

$$\mathbb{C}_e \operatorname{sym}(\nabla u - P) + \mathbb{C}_c \operatorname{skew}(\nabla u - P) - \mathbb{C}_{\text{micro}} \operatorname{sym} P = 0$$

whose symmetric part gives the two following equations

$$\mathbb{C}_e \operatorname{sym}(\nabla u - P) = \mathbb{C}_{\text{micro}} \operatorname{sym} P \quad (2)$$

and

$$\mathbb{C}_e \operatorname{sym} \nabla u = (\mathbb{C}_{\text{micro}} + \mathbb{C}_e) \operatorname{sym} P \quad \Leftrightarrow \quad \operatorname{sym} P = (\mathbb{C}_{\text{micro}} + \mathbb{C}_e)^{-1} \mathbb{C}_e \operatorname{sym} \nabla u \quad (3)$$

Micro-macro homogenization formula (tensor harmonic mean)

Replacing (2) and (3) in (1), we obtain

$$\text{Div}\left[\mathbb{C}_{\text{micro}}\,\text{sym}\,P\right]=0 \quad \Leftrightarrow \quad \text{Div}\left[\mathbb{C}_{\text{micro}}\left(\mathbb{C}_{\text{micro}}+\mathbb{C}_e\right)^{-1}\mathbb{C}_e\text{sym}\nabla u\right]=0.$$

This formula is impossible in classical micromorphic elasticity (needs specific split in sym and skew).

In the tetragonal case:

$$\mu_{\text{macro}}=\frac{\mu_e\,\mu_{\text{micro}}}{\mu_e+\mu_{\text{micro}}}, \qquad \mu^*_{\text{macro}}=\frac{\mu^*_e\,\mu^*_{\text{micro}}}{\mu^*_e+\mu^*_{\text{micro}}},$$

$$2\mu_{\text{macro}}+3\lambda_{\text{macro}}=\frac{\left(2\mu_e+3\lambda_e\right)\left(2\mu_{\text{micro}}+3\lambda_{\text{micro}}\right)}{2\left(\mu_e+\mu_{\text{micro}}\right)+3\left(\lambda_e+\lambda_{\text{micro}}\right)}.$$

- Barbagallo G., Madeo A., d'Agostino M.V., Abreu R., Ghiba I.D., and Neff P. Transparent anisotropy for the relaxed micromorphic model: macroscopic consistency conditions and long wave length asymptotics. **International Journal of Solids and Structures**,Vol. 120, 7-30, 2017.

Micro-macro homogenization formula (tensor harmonic mean)

- Limit for large, **engineering-size samples** $\longrightarrow$ **linear elasticity** with $\mathbb{C}_{\mathbf{macro}}$
- Limit for **unit-cell structure** $\longrightarrow$ **linear elasticity** with $\mathbb{C}_{\mathbf{micro}}$
- Both $\mathbb{C}_{\mathbf{micro}}$ and $\mathbb{C}_{\mathbf{macro}}$ are available:

  $\mathbb{C}_{\mathbf{macro}} \longleftarrow$ periodic homogenization or Bloch-Floquet (long wave-length response, tangents of the acoustic curves, infinite periodic medium)

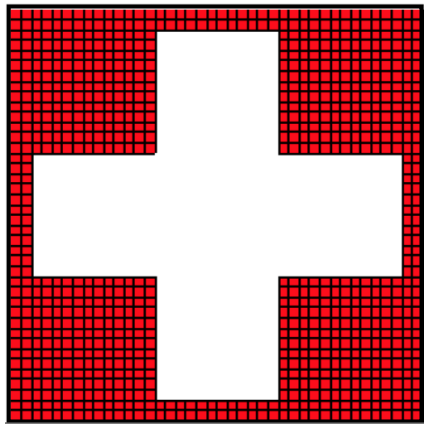
  $\mathbb{C}_{\mathbf{micro}} \longleftarrow$ calculation of the stiffness of the unit-cell

$$\mathbb{C}_{\text{micro}} \xrightarrow[\substack{\text{taking into account}\\ \text{the actual specimen}\\ \text{size}}]{\substack{\text{relaxed micromorphic}\\ \downarrow\\ \text{interpolate}}} \mathbb{C}_{\text{macro}}$$

Identification of $\mathbb{C}_{\text{macro}}$ - periodic homogenization

Periodic boundary conditions mimic an infinite number of unit cells.

Calculation of macroscopic stiffness $\mathbb{C}_{\text{macro}}$ by **numerical homogenization**.



λ_{macro}	μ_{macro}	μ^*_{macro}
1.738	5.895	0.620
[GPa]	[GPa]	[GPa]

$$\frac{1}{2}\langle \mathbb{C}_{\text{macro}}\,\bar{E},\bar{E}\rangle\,|\Omega_c| :=$$

$$\inf\left\{\int_{\xi\in\Omega_c}\frac{1}{2}\left\langle \mathbb{C}(\xi)\ \text{sym}\left(\nabla_\xi v(\xi)+\overline{E}\right),\text{sym}\left(\nabla_\xi v(\xi)+\overline{E}\right)\right\rangle d\xi \;\middle|\; v\in\mathcal{C}^\infty\left(\Omega_c,\mathbb{R}^3\right)\ \text{is periodic}\right\},$$

where $\mathbb{C}(\xi)$ is the elasticity tensor of the aluminum phase and air depending on the position of ξ in the unit-cell and $\overline{E}=\text{sym}\nabla u(x)$.

Periodic stiffness $\mathbb{C}_{\text{macro}}$ is **independent** of the size and shape of unit-cell.

Identification of $\mathbb{C}_{\text{micro}}$ - stiffest tetragonal unit-cell under affine Dirichlet boundary conditions (KUBC)

Calculation of microscopic stiffness $\mathbb{C}_{\text{micro}}$ by **numerical homogenization**.
Energy minimization problem

$$\int_\Omega W\left(\nabla u, P, \operatorname{Curl} P\right) dm \quad \longrightarrow \quad \min\ (u, P) \in H^1(\Omega) \times H(\operatorname{Curl};\, \Omega)\,,$$

whit boundary conditions

$$u|_{\partial\Omega}(x) = \overline{B} \cdot x, \quad \overline{B} \in \mathbb{R}^{3\times 3}, \qquad \nabla u|_{\partial\Omega}(x) \cdot \tau_{1,2} = P|_{\partial\Omega}(x) \cdot \tau_{1,2},$$

$\tau_{1,2}$ are tangent vectors to $\partial\Omega$. The minimization problem

$$\inf_{(u,P)} \int_{x\in\Omega} W\left(\nabla u, P, \operatorname{Curl} P\right) dx$$

can be bounded above by taking $P(x) = \nabla u(x)$. Indeed

$$\begin{aligned}\inf_{(u,P)} \int_{x\in\Omega} W\left(\nabla u, P, \operatorname{Curl} P\right) dx &\leq \inf_{u} \int_{x\in\Omega} W\left(\nabla u, \nabla u, 0\right) dx \\ &= \inf_{u} \int_{x\in\Omega} \frac{1}{2} \left\langle \mathbb{C}_{\text{micro}} \operatorname{sym} \nabla u(x)\,, \operatorname{sym} \nabla u(x) \right\rangle dx.\end{aligned}$$

Identification of $\mathbb{C}_{\text{micro}}$ - stiffest tetragonal unit-cell under affine Dirichlet boundary conditions (KUBC)

Therefore, the maximal possible stored elastic energy of the relaxed micromorphic model over an arbitrary window $\widetilde{\Omega} \subset \Omega$ is

$$\inf_{u} \int_{x \in \widetilde{\Omega}} \frac{1}{2} \left\langle \mathbb{C}_{\text{micro}} \operatorname{sym} \nabla u(x), \operatorname{sym} \nabla u(x) \right\rangle dx, \qquad u|_{\partial\Omega}(x) = \overline{B} \cdot x,$$

Taking $\widetilde{\Omega}$ as the unit cell $V(x)$

$$\frac{1}{|V(x)|} \int_{\xi \in V(x)} \nabla_\xi u(x+\xi)\, d\xi = \frac{1}{|V(x)|} \int_{\xi \in V(x)} \overline{B}\, d\xi = \overline{B},$$

and symmetrizing

$$\overline{\varepsilon} = \frac{1}{|V|} \int_{\xi \in V(x)} \varepsilon(x+\xi)\, d\xi = \operatorname{sym} \overline{B} = \overline{E}.$$

From convexity for homogeneous boundary conditions $\overline{v}(\xi) = \overline{B} \cdot \xi$

$$\inf \left\{ \int_{\xi \in V(x)} \frac{1}{2} \left\langle \mathbb{C}_{\text{micro}} \operatorname{sym} \nabla_\xi \overline{v}(\xi), \operatorname{sym} \nabla_\xi \overline{v}(\xi) \right\rangle d\xi \;\middle|\; \overline{v} : V(x) \to \mathbb{R}^3,\ \overline{v}|_{\partial V(x)}(\xi) = \overline{B} \cdot \xi \right\}$$

$$= \frac{1}{2} \left\langle \mathbb{C}_{\text{micro}} \operatorname{sym} \overline{B}, \operatorname{sym} \overline{B} \right\rangle |V(x)| = \frac{1}{2} \left\langle \mathbb{C}_{\text{micro}} \overline{E}, \overline{E} \right\rangle |V(x)|.$$

Identification of $\mathbb{C}_{\text{micro}}$ - stiffest tetragonal unit-cell under affine Dirichlet boundary conditions (KUBC)

We demand that the fine-scale energy should equal the coarse-scale energy over the same domain $V(x)$, under the same affine boundary conditions $\overline{B} \cdot \xi$ and for the same material. This implies that for all $\overline{B} \in \mathbb{R}^{3\times 3}$:

$$\underbrace{\frac{1}{2}\left\langle \mathbb{C}_{\text{micro}}\,\overline{E},\overline{E}\right\rangle |V(x)|}_{\substack{\text{coarse scale micromorphic}\\ \text{upper energy limit}}} \geq \underbrace{\inf\left\{\int_{\xi\in V(x)} \frac{1}{2}\left\langle \mathbb{C}\left(\xi\right)\left(\text{sym}\nabla_{\xi} v\left(\xi\right)+\overline{E}\right),\text{sym}\nabla_{\xi} v\left(\xi\right)+\overline{E}\right\rangle d\xi \;\middle|\; v\in \mathcal{C}_0^{\infty}\left(V(x),\mathbb{R}^3\right)\right\}}_{\text{fine-scale linear elastic energy}}. \tag{4}$$

According to the **Hill-Mandel energy equivalence**, we can **define** a **unique apparent stiffness** tensor $\mathbb{C}^{V}_{\text{KUBC}}$, independent of $\overline{E}$,

$$\underbrace{\inf\left\{\int_{\xi\in V(x)} \frac{1}{2}\left\langle \mathbb{C}\left(\xi\right)\left(\text{sym}\nabla_{\xi} v\left(\xi\right)+\overline{E}\right),\text{sym}\nabla_{\xi} v\left(\xi\right)+\overline{E}\right\rangle d\xi \;\middle|\; v\in \mathcal{C}_0^{\infty}\left(V(x),\mathbb{R}^3\right)\right\}}_{\text{fine-scale linear elastic energy}} = \frac{1}{2}\left|V\left(x\right)\right|\left\langle \mathbb{C}^{V}_{\text{KUBC}}\,\overline{E},\overline{E}\right\rangle. \tag{5}$$

Identification of $\mathbb{C}_{\text{micro}}$ - stiffest tetragonal unit-cell under affine Dirichlet boundary conditions (KUBC)

We demand that the fine-scale energy should equal the coarse-scale energy over the same domain $V(x)$, under the same affine boundary conditions $\overline{B} \cdot \xi$ and for the same material. This implies that for all $\overline{B} \in \mathbb{R}^{3\times 3}$:

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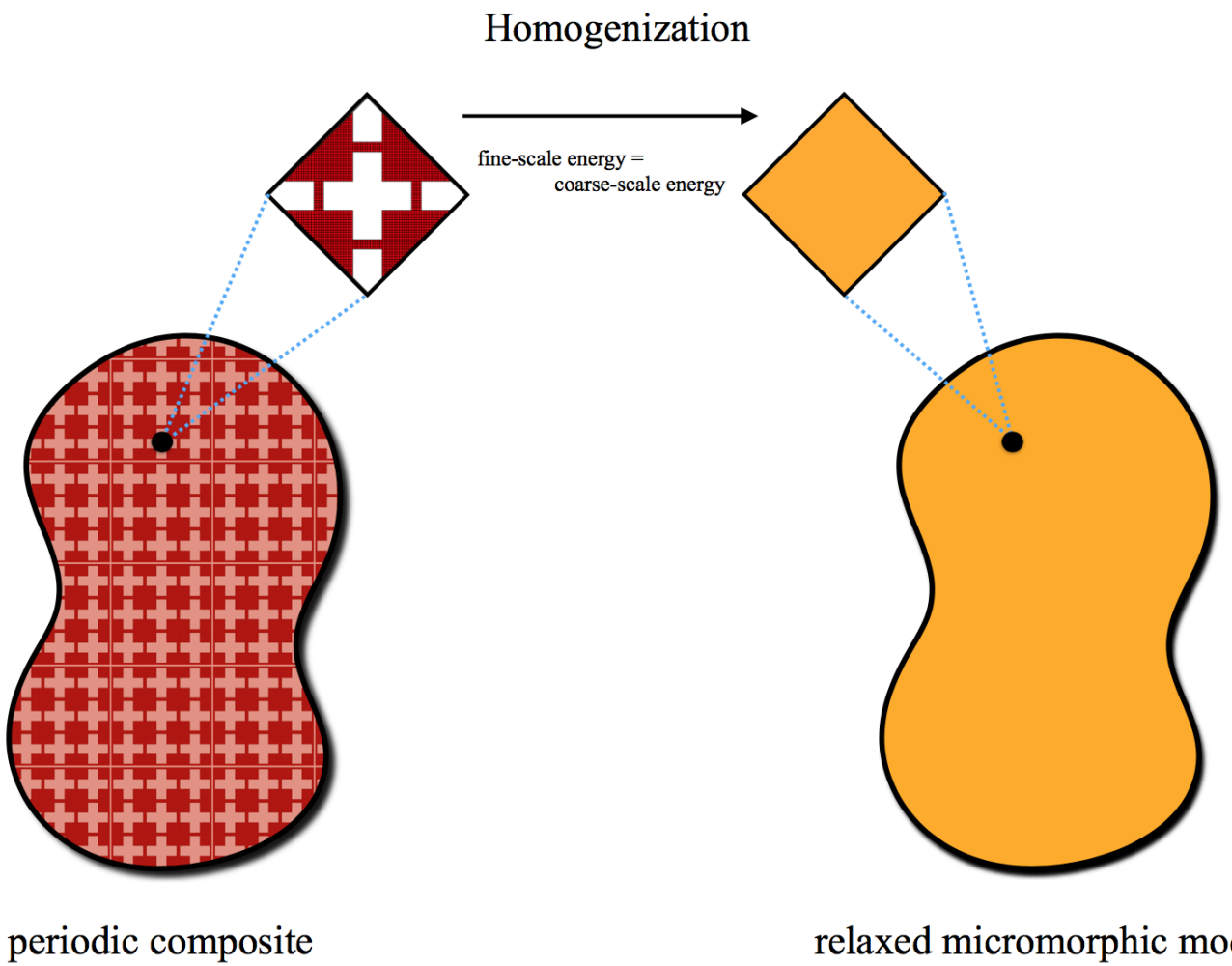
Identification of $\mathbb{C}_{\text{micro}}$ - stiffest tetragonal unit-cell under affine Dirichlet boundary conditions (KUBC)



Identification of $\mathbb{C}_{\text{micro}}$ - stiffest tetragonal unit-cell under affine Dirichlet boundary conditions (KUBC)

- **Extended Neumann's principle** allows us to determine the tensor $\mathbb{C}_{\text{micro}}$ by requiring

$$\forall\, \overline{E} \in \operatorname{Sym}(3) \;:\; \left\langle \mathbb{C}_{\text{micro}}\, \overline{E}, \overline{E} \right\rangle \geq \left\langle \mathbb{C}^{V}_{\text{KUBC}}\, \overline{E}, \overline{E} \right\rangle, \tag{6}$$

  and

$$\left\langle \mathbb{C}^{*}_{\text{micro}}\, \overline{E}, \overline{E} \right\rangle \geq \left\langle \mathbb{C}_{\text{micro}}\, \overline{E}, \overline{E} \right\rangle$$

  for any other possible tensor $\mathbb{C}^{*}_{\text{micro}}$ which verifies (6).

- **Dirichlet-stiffness** $\mathbb{C}_{\text{micro}}$ **depends** on the size and shape of unit cell.
  In general, bigger unit cell $\rightsquigarrow$ less stiff.

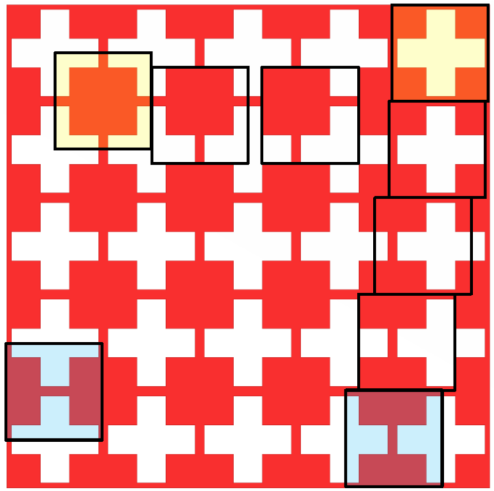
- $\mathbb{C}_{\text{micro}}$ is **defined unambiguously** (**Löwner matrix supremum**). Indeed, for all $(x, y, z)^T \in \mathbb{R}^3$

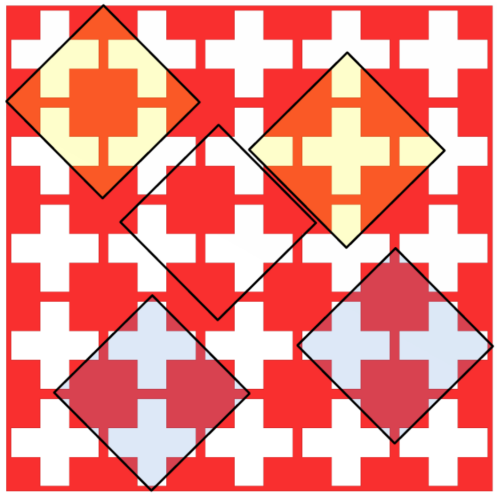
$$\left\langle \begin{pmatrix} 2\widehat{\mu}+\widehat{\lambda} & \widehat{\lambda} & 0 \\ \widehat{\lambda} & 2\widehat{\mu}+\widehat{\lambda} & 0 \\ 0 & 0 & \widehat{\mu}^{*} \end{pmatrix}_{\mathbb{C}_{\text{micro}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \geq \left\langle \begin{pmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu^{*} \end{pmatrix}_{\mathbb{C}^{V}_{\text{KUBC}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle, \tag{7}$$

  where (x, y, z) represents $\left(\overline{E}_{11}, \overline{E}_{22}, \overline{E}_{12}\right)$.

Identification of $\mathbb{C}_{\text{micro}}$ - stiffest tetragonal unit-cell under affine Dirichlet boundary conditions (KUBC)

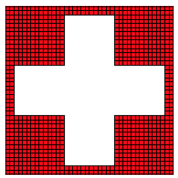
Many possible unit-cells: size, shape, symmetry,...



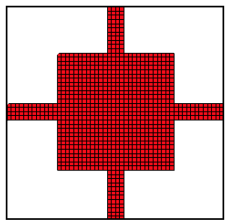


	λ_{hom}	μ_{hom}	μ^*_{hom}
(a)	4.37	6.242	**8.332**
(b)	2.125	5.899	2.264
(c)	5.270	**8.927**	4.042
(d)	**5.981**	6.254	4.96
		$\mathbb{C}_{\text{micro}}$	
	[GPa]	[GPa]	[GPa]

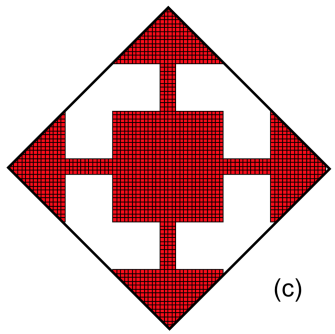
Neumann's principle: restriction to tetragonal unit-cells



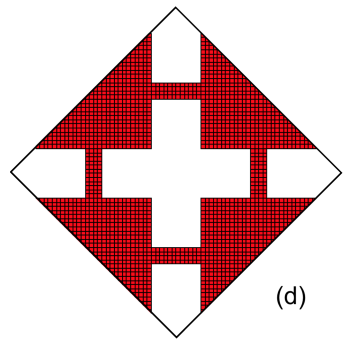
(a)



(b)

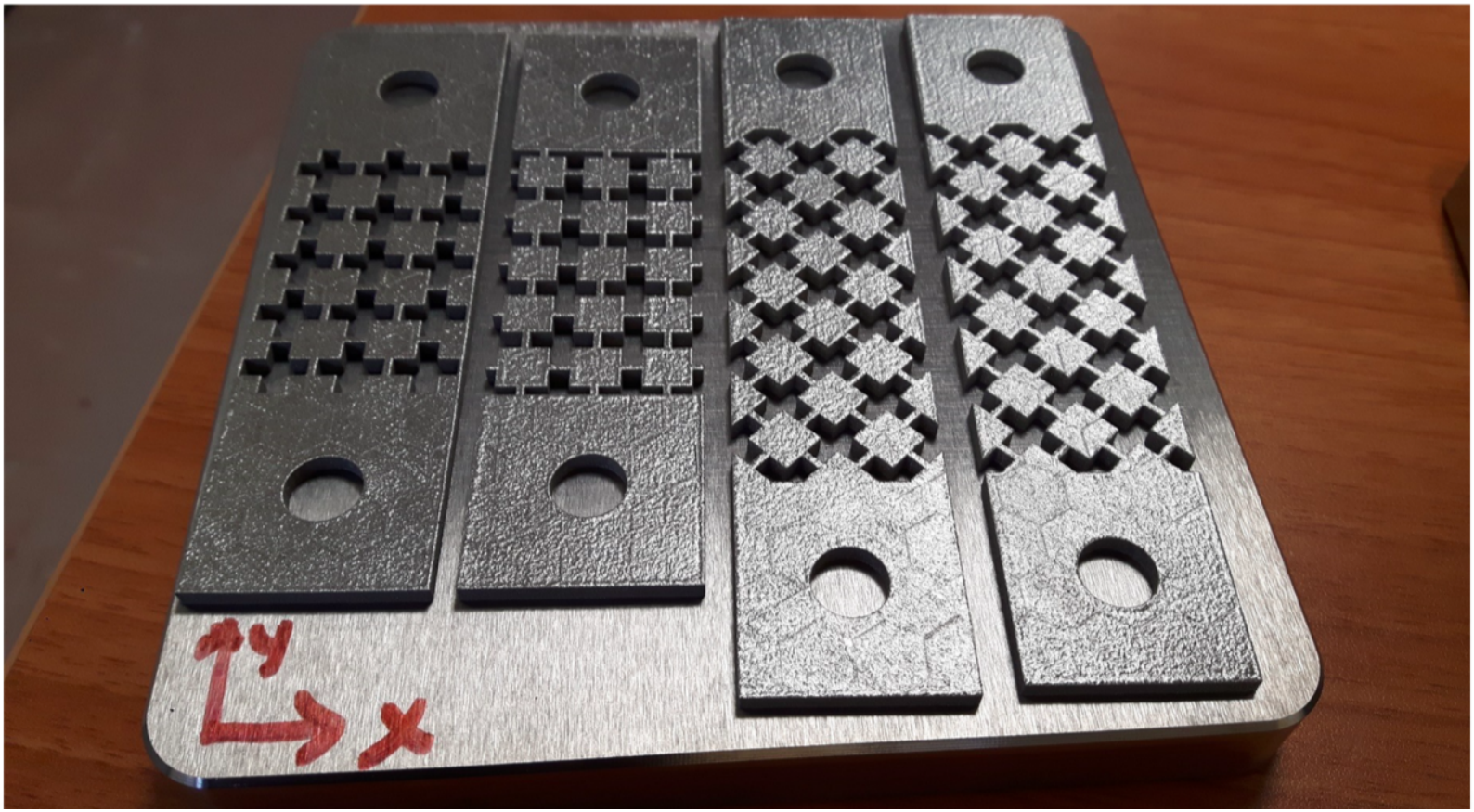


(c)



(d)

Identification of $\mathbb{C}_{\text{micro}}$ - stiffest tetragonal unit-cell under affine Dirichlet boundary conditions (KUBC)



Picture by M. Collet, Ecole central Lyon.

Identification of $\mathbb{C}_{\text{micro}}$ - stiffest tetragonal unit-cell under affine Dirichlet boundary conditions (KUBC)

Denoting the four KUBC-entries by $\mu_i, \lambda_i, \mu_i^*$, $i = 1, 2, 3, 4$, respectively, the question is how to obtain the values $\widehat{\mu}, \widehat{\lambda}, \widehat{\mu}^*$ such that

$$\left\langle \begin{pmatrix} 2\widehat{\mu}+\widehat{\lambda} & \widehat{\lambda} & 0 \\ \widehat{\lambda} & 2\widehat{\mu}+\widehat{\lambda} & 0 \\ 0 & 0 & \widehat{\mu}^* \end{pmatrix}_{\mathbb{C}_{\text{micro}}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle \geq \left\langle \begin{pmatrix} 2\mu_i+\lambda_i & \lambda_i & 0 \\ \lambda_i & 2\mu_i+\lambda_i & 0 \\ 0 & 0 & \mu_i^* \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle,$$

$i = 1, 2, 3, 4$, is always verified. Since μ^* sits on the diagonal, we must necessarily have that

$$\mu^* \geq \mu_i^* \qquad \forall\, i = 1, 2, 3, 4.$$

Therefore, the problem is **reduced to the 2×2 block**: for all $(x, y)^T \in \mathbb{R}^2$

$$\left\langle \begin{pmatrix} 2\widehat{\mu}+\widehat{\lambda} & \widehat{\lambda} \\ \widehat{\lambda} & 2\widehat{\mu}+\widehat{\lambda} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle \geq \left\langle \begin{pmatrix} 2\mu_i+\lambda_i & \lambda_i \\ \lambda_i & 2\mu_i+\lambda_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle, \quad i = 1, 2, 3, 4.$$

According to the **Sylvester-criterion** for the difference of positive definite tensors, we must have $\widehat{\mu} \geq \mu_i$ and $\widehat{\mu} + \widehat{\lambda} \geq \mu_i + \lambda_i$ for $i = 1, 2, 3, 4$. We choose

$$\widehat{\mu} := \max_i \{\mu_i\}, \quad \widehat{\mu}^* := \max_i \{\mu_i^*\}, \quad \widehat{\mu} + \widehat{\lambda} := \max_i \{\mu_i + \lambda_i\}, \quad \widehat{\lambda} := \max_i \{\mu_i + \lambda_i\} - \widehat{\mu}.$$

This determines μ, μ^*, λ **uniquely**.

Fitting procedure

Cut-off frequencies for

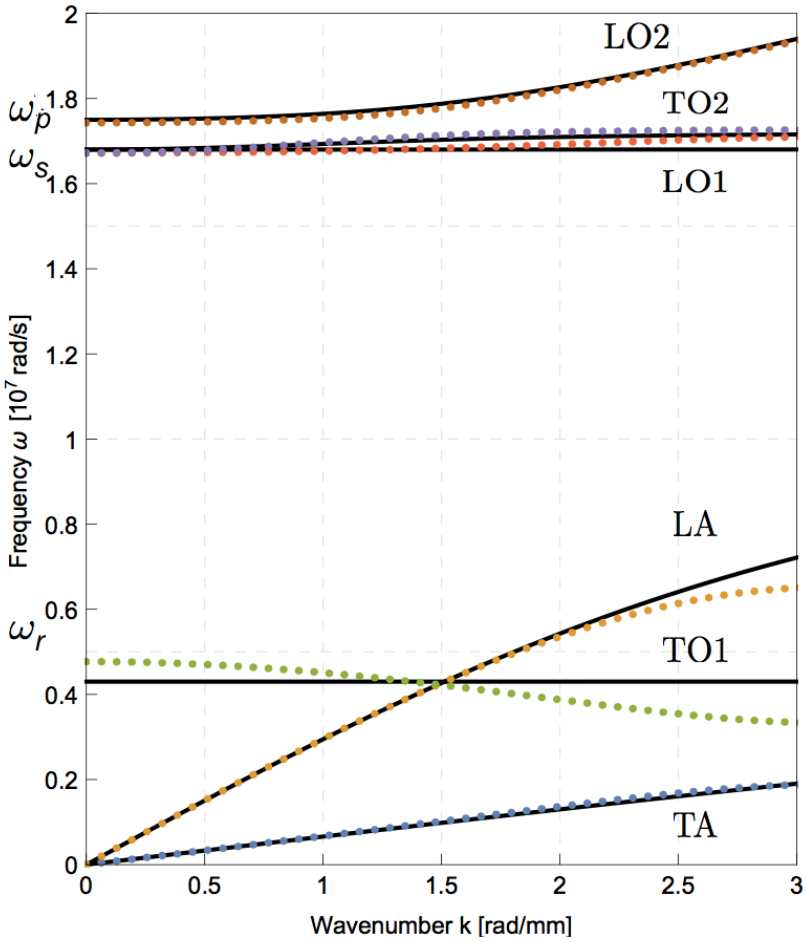
$$\det D(k,\omega) = 0 \quad \text{at} \quad k = 0$$

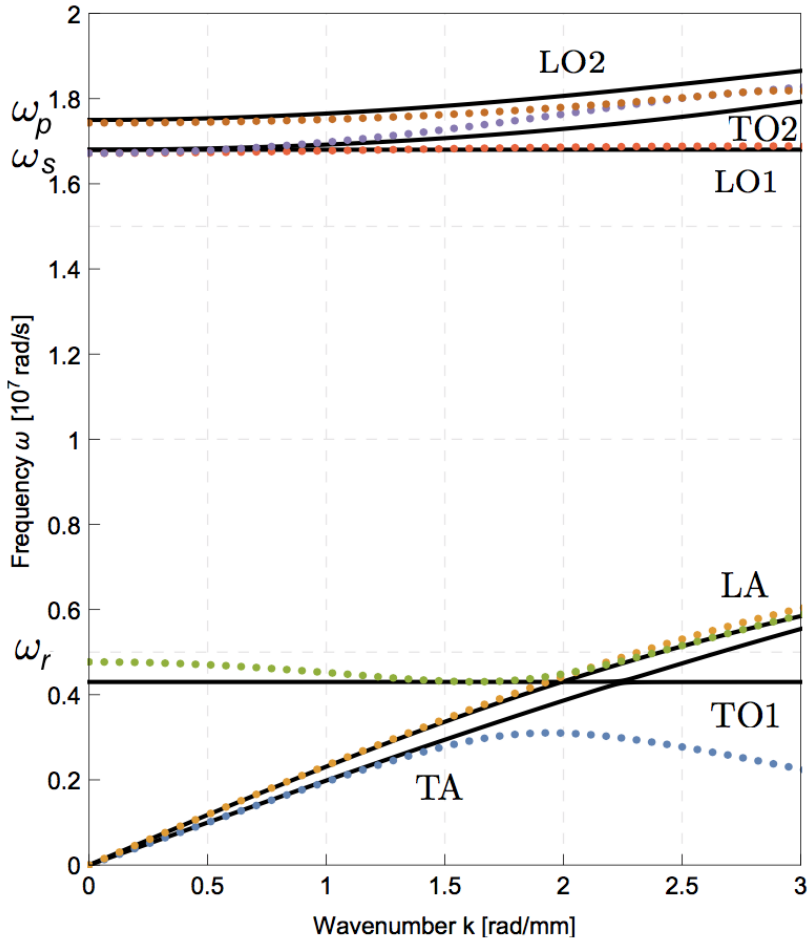
are

$$\omega_r = \sqrt{\frac{\mu_c}{\eta_2}}, \qquad \omega_s = \sqrt{\frac{\mu_e + \mu_{\text{micro}}}{\eta_1}},$$

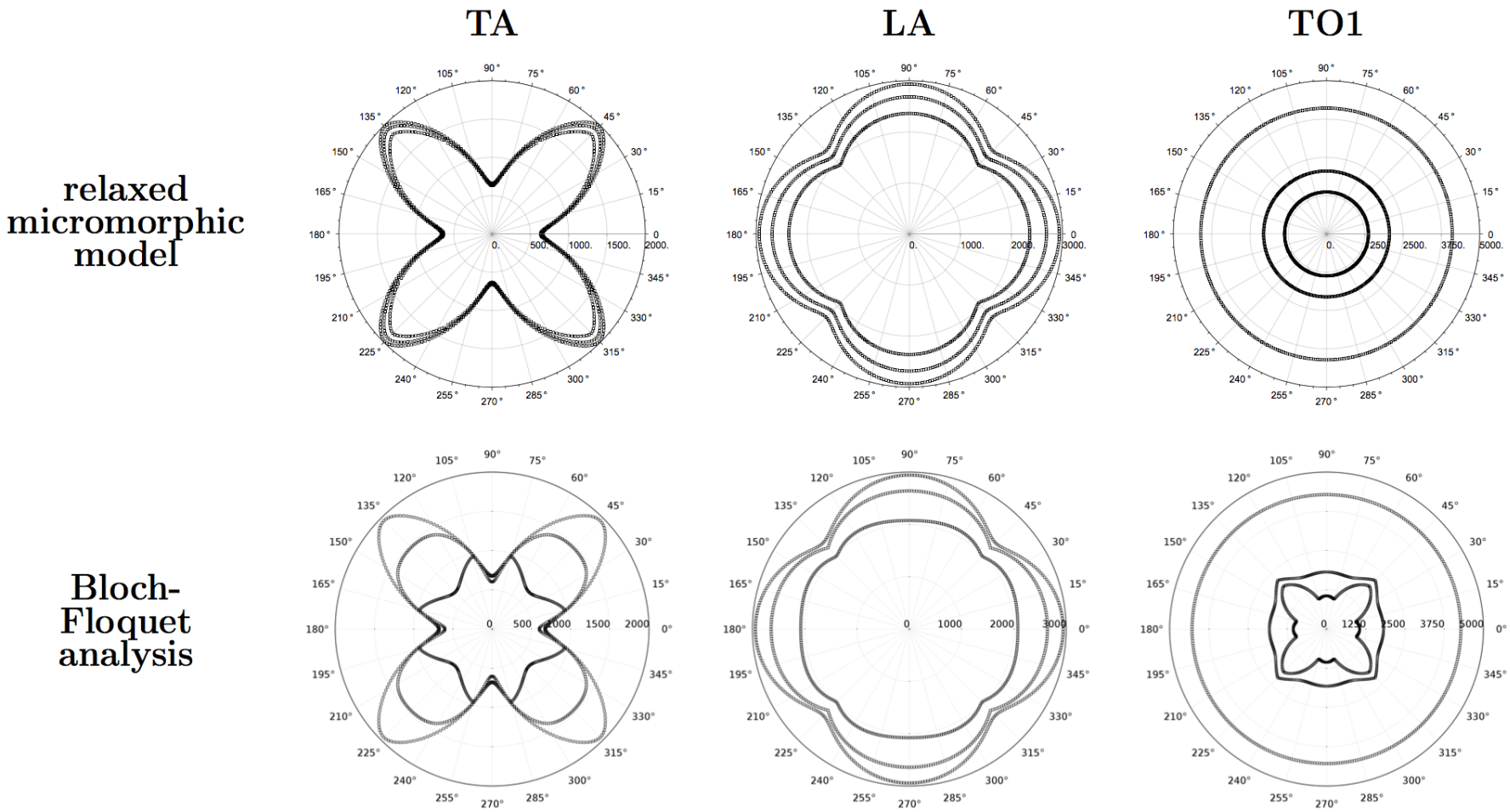
$$\omega_s^* = \sqrt{\frac{\mu_e^* + \mu_{\text{micro}}^*}{\eta_1^*}}, \qquad \omega_p = \sqrt{\frac{\mu_e + \mu_{\text{micro}} + \lambda_e + \lambda_{\text{micro}}}{\eta_1 + \eta_3}}.$$

Dispersion and anisotropy in tetragonal metamaterials



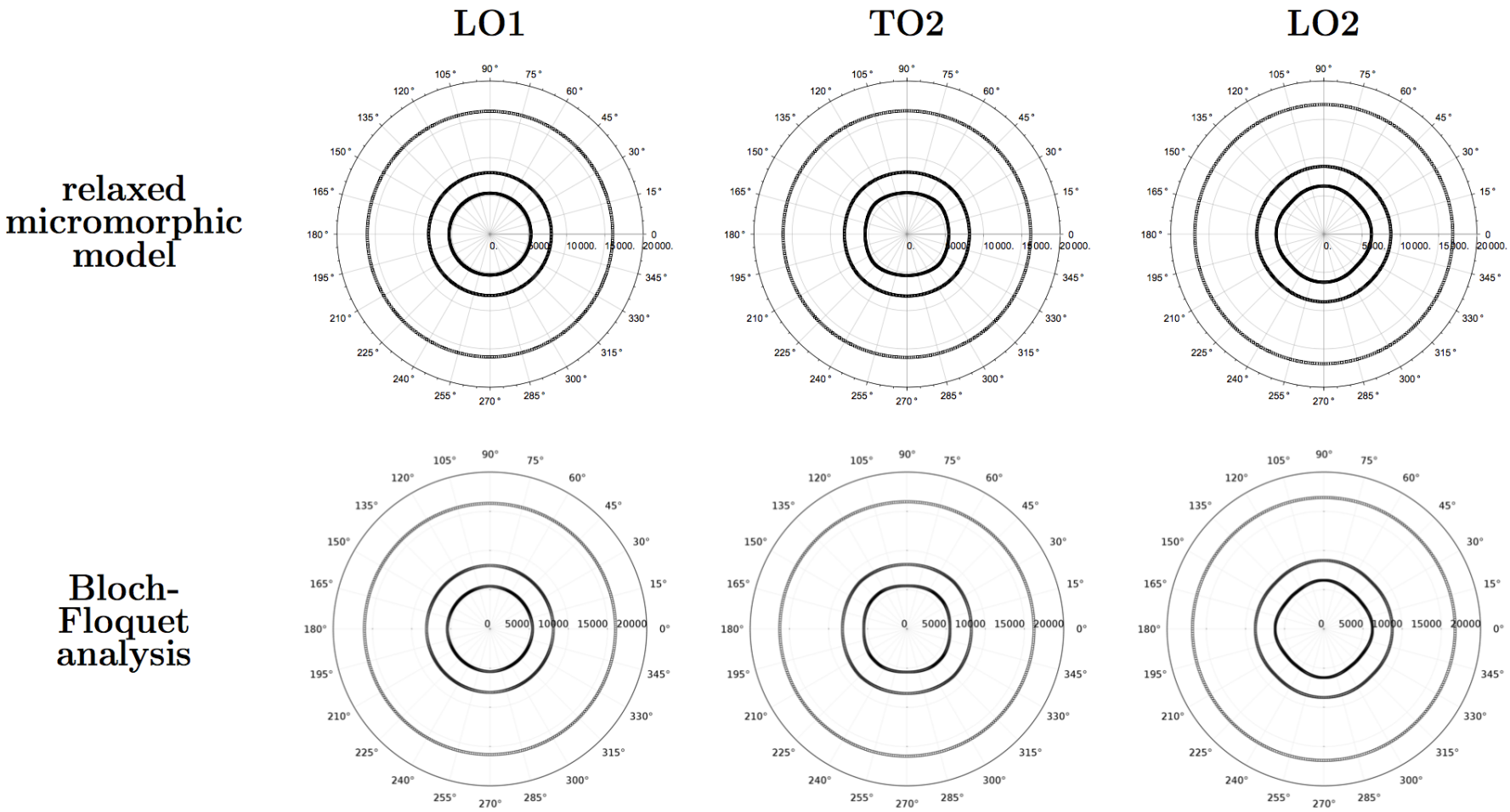


Phase velocity $\frac{\omega(k)}{k}$: polar plots



Phase velocity $\frac{\omega(k)}{k}$: polar plots as a function of the direction of wave propagation $\widehat{\boldsymbol{k}}$. The plotted curves have been calculated for three different values of the wave number k.

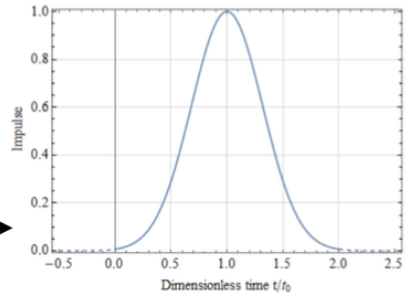
Phase velocity $\frac{\omega(k)}{k}$: polar plots

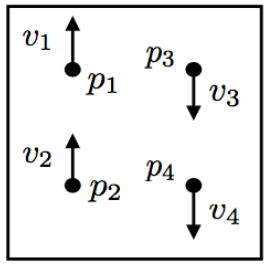


Phase velocity $\frac{\omega(k)}{k}$: polar plots as a function of the direction of wave propagation $\widehat{\boldsymbol{k}}$. The plotted curves have been calculated for three different values of the wave number k.

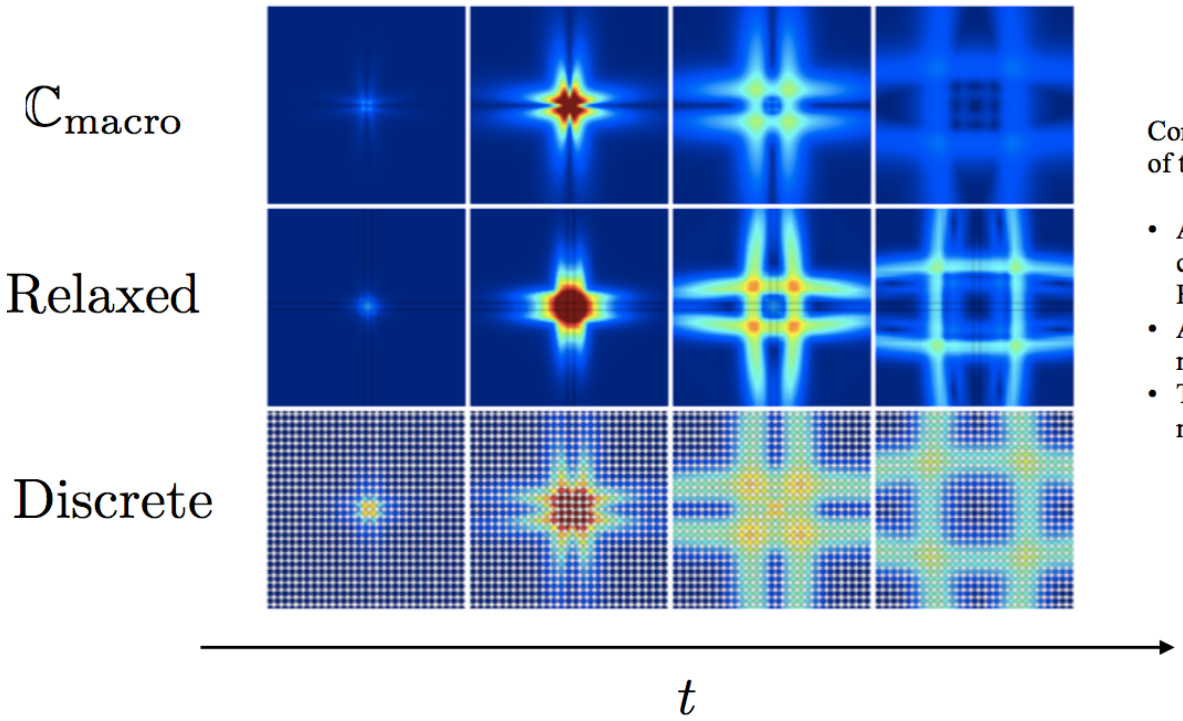
Impulses

$$\varphi(x,t) = \mathcal{I}(t) \sum_{j=1}^{4} v_j \delta(x - p_j)$$





Shear loading



Comparison of the propagation of the same impulse in

- A classical Cauchy continuum medium with Hooke tensor $\mathbb{C}_{\text{macro}}$
- A relaxed micromorphic medium
- The microstructure metamaterial

Conclusions

- Our framework allows to **describe anisotropic dispersion in metamaterials** while **avoiding unnecessary complexifications** related to the introduction of elastic tensors of order higher than four, as it is the case for higher gradient elasticity (exception: hexagonal symmetry).
- The model **describes dispersion and anisotropy**, not only for the first two acoustic modes, but also **for other modes at higher frequencies**.
- Complete **a priori identification** of static material parameters by numerical homogenization: $\mathbb{C}_{\text{macro}}$ (**periodic** bc on **any unit-cell**), $\mathbb{C}_{\text{micro}}$ (**affine Dirichlet** on **stiffest unit-cell**).
- Perspective: modeling metamaterials in a simplified framework, so providing a concrete possibility for the **design of complex metastructures**.
- Improved calibration for high wave numbers (small wavelength) based on new non local inertia term $\operatorname{Curl} P_{,t}$ will be investigated in **future works**.

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Thank you for your attention!

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